Decision Analytics Journal 10 (2024) 100402

Contents lists available at ScienceDirect

Decision Analytics Journal

journal homepage: www.elsevier.com/locate/dajour

A Fuzzy Fractional Power Series Approximation and Taylor Expansion for Solving Fuzzy Fractional Differential Equation

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ARTICLE INFO

Keywords:
Fuzzy fractional differential equation
Fuzzy Caputo fractional derivative
Fuzzy fractional Taylor's theorem
Dynamical systems
Power series approximation

ABSTRACT

Fuzzy fractional differential has the strength to capture the senses of memory and uncertainty simultaneously involved in dynamical systems. However, a solution for fuzzy fractional differential equations is not always found regularly. This paper discusses a numerical solution approach for the fuzzy fractional differential equation using power series approximation with a fuzzy fractional counterpart of Taylor's theorem. Caputo's definition of the fractional derivative and generalized Hukuhara difference are used to describe the fuzzy differential equation in this paper. Utilization of the generalized Hukuhara difference for the fuzzy valued function ensures the uniqueness and boundedness of the fuzzy solution in parametric form.

1. Introduction

The importance of the Power series is to analyse different complex functions and evaluate its solutions in various numerical methods. Solve the integer, fractional and nonlinear differential equations problems and determine the solution region. Consistently, the fuzzy set and fuzzy fractional differential equations with Caputo fractional derivative operation [1] have been applied in this paper. This section discusses the background and motivation of this study in detail.

1.1. Fractional differential equations

Fractional differential equations (FDEs) consist of fractional derivatives having non-local characteristics. The non-local behaviour of the fractional derivative can be interpreted through its memory-carrying ability in dynamical models. Thus, fractional differential equations have a wide spectrum of application domains in today's scientific and industrial fields. In the present century's research contemporary, fractional differential is regarded as a robust mathematical tool for modelling concerning non-local or non-Markovian real-world processes, such as viscoelasticity, signal processing, and diffusion with long-range memory, control processing, fractional stochastic systems, and allometry in biology and ecology [2–16]. Though the application-based study of fractional calculus is present century contemporary, the root of the theoretical advancement of fractional calculus was laid by the works of renowned mathematicians like Leibniz, Euler, and Liouville who encountered integrals and derivatives with non-integer orders while working through a variety of mathematical problems, and these encounters are where the first clues of fractional calculus may be discovered. The beginning of more organized development, however, did not occur until the 19th century. Several kinds of fractional derivatives are defined in the literature among which Riemann–Liouville and Caputo derivatives are the most widely used fractional derivatives. Fractional differential equations are solved using various numerical techniques [17–25] like the Trapezoidal method, Predictor–Corrector method, Spectral method, etc.

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https://doi.org/10.1016/j.dajour.2024.100402
Received 24 October 2023; Received in revised form 16 January 2024; Accepted 18 January 2024
Available online 22 January 2024

1.2. Fuzzy set and its background

In crisp set theory, the basic concept is an element is either a member of the set or not a member of the set; i.e., $x \in A$ or $x \notin A$. Although, in the real world the membership of an element is not always crisp. The fuzzy set theory is considered the basis of the identification of belongingness and non-belongingness. In 1965, Professor Lotfi A. Zadeh first introduced the Fuzzy set [26] concept, where every element of the set has a unique membership value. The membership values denote the degree of belongness of the element in the set. After that, the fuzzy set has several extensions over time. The fuzzy set can be represented in an interval called $\alpha-$cut of a fuzzy set [27]. Singh, P. et al. [28] use $\alpha-$cut method of the fuzzy set to construct the Malaria model and solve. Bhattacharya, P. P. et al. [29] used Triangular Fuzzy Set (TFS) to analysis environmental pollution. The fuzzy differential equation was solved using fuzzy Caputo's differentiability by Akram, M. et al. [30]. Similarly, Ullah, A. et al. [31] applied fuzzy Yang transform to solve second order differential equations.

Fuzzy sets used in numerous fields, like, Chakraborty, S. et al. [32] applied in health care west management system, Ghorui, N. et al. [33] used in location selection for shopping malls and Choudhury, S. et al. [34] evaluated the risk factors opposed to COVID 19 using different MCDM methodologies. To solve differential equations fuzzy sets play crucial roles. Mondal, S. P. et al. [35] used triangular fuzzy set to solve the second order linear fuzzy differential equations. To solve the linear differential equations appalled intuitionistic fuzzy uncertainty by Mondal, S. P. et al. [36] and check the existence & stability of the solution of the differential equation used fuzzy uncertain environment by Mondal. S. P. et al. [37].

1.3. Fuzzy differential equations

Zadeh, L. A. [26] innovated the philosophy of fuzziness in 1965, which put many intermediate grey scopes between the "belongs to" and "not belongs to" approach in the classical set theory. Chang and Zadeh [38] introduced the notion of fuzzy derivatives to describe differential equations under fuzzy uncertainty. There are several proposed types of fuzzy derivatives, which are helpful in resolving practical engineering and scientific problems. Buckley et al. [39] solve the fuzzy first order differential equation. Initial explanations of fuzzy mappings and the distinctions between various fuzzy mapping approaches were provided by Dubois and Prade [40–42] in their consecutive works. Additionally, they talked about the fuzzy differentiation of fuzzy valued functions. Differentiability and integrability characteristics were detailed by Kaleva, O. [43]. Puri and Ralescu [44] extended the differentiability of the set-valued function introducing Hukuhara derivative for the fuzzy-valued functions. Their works were followed by several consecutive investigations [43–45] on differential equations with fuzzy initial conditions and coefficients. The use of Hukuhara derivatives has the limitation that fuzzy solutions eventually become unbounded. Strongly generalized-Hukuhara derivative and generalized-Hukuhara differentiability notions were proposed in [46,47] and were demonstrated to be superior to the Hukuhara derivative. However, the drawback of employing these derivatives is that the uniqueness of the solution is lost. The modified Hukuhara derivative (mth derivative), recently introduced in [48] providing a unique and bounded fuzzy solution.

Tudu et al. [49] solve non-homogenous linear differential equations using fuzzy numbers. Keshavarz et al. [50] developed mathematical models to solve the first and second order fuzzy differential equations and applied in Drug distribution in the Human body, Newton's low of cooling, harmonic oscillation problems, etc. The fuzzy differential equation also applied in biological model by Routaray et al. [51], tumor growth design in the human body applying generalized Hukuhara derivative by Khaliq et al. [52] and COVID-19 prediction model by Padmapriya et al. [53]. Cazarez-Castro et al. [54] applied this fuzzy differential equation (FDE) in engineering field, Nadeem et al. [55] use the FDE to analyse the Heat transformation process between two Plates and Qiu et al. [56] applied FDE in fuzzy numbers quotient space. The malaria model is developed and solved using FDE by Singh et al. [28].

1.4. Fuzzy fractional differential equation

Suppose a memory concerned dynamical system is taken into a mathematical model under uncertain phenomena to deal with, the theory of fuzzy fractional differential equation is one of the effective and essential mathematical tools in this concern. In that context, fractional differential equation stands for memory-based dynamics through its non-local characteristic while fuzzy logic refers uncertainty in mathematical framework. Agarwal et al. [57] first considered fractional differential equation under fuzzy uncertainty in which he used Riemann–Liouville definition for fractional derivative. Following his footstep, the genre of fuzzy fractional differential equations was established in a last decade. In this context, Hoa et al. [58] discussed fuzzy fractional differential equations in terms of Caputo fractional derivatives. For the uniqueness of the solution of uncertain fractional differential equations, Lupulescu et al. [59] established generalized Hukuhara fractional differentiability under uncertainty. To deal with manifestation of the solution, fuzzy Laplace transformation is very significant mathematical tool which was introduced by Allahviranloo and Ahmadi [60]. Salahshour and Allahviranloo [61] used the notion of fuzzy Laplace transformation to solve fuzzy fractional differential equations. Several more theoretical advancements [62–68] were done in this regard. However, explicit solutions for fuzzy fractional differential equations cannot be derived in an easy manner. In those cases, numerical approaches do the job. In literature, we find a few innovative techniques to find numerical solutions for the fuzzy fractional differential equations.

This paper exhibits a new approach to get the numerical solution by taking fuzzy fractional power series approximation due to fuzzy fractional analogue of the Taylor's theorem [69–71]. Here, the fuzzy fractional Taylor's expansion [72,73] is done under fuzzy Caputo fractional derivative [74]. The convergence of the fuzzy power series containing fraction terms is proved. Table 1 summarizes the contributions of this paper and the related existing works.

1.5. Background of Fuzzy fractional power series

This section discussed the fuzzy power series (FPS) and its application. The fuzzy power series [1] representation shows the efficiency of the representative and computational complexity of the model. This study deals with solutions of fuzzy fractional differential equations using fuzzy Caputo fractional derivative operator [42] and fractional Taylor's theorem [77]. Use the generalized Hukuhara derivative and the generalized Hukuhara difference to represent the fuzzy power series solution in this study. This power series expansion use in enormous field, like, Shahidi, M. et al. [78] use power series expansion to solve fuzzy differential equations, Alomari, A. K. et al. [79] solve the nonlinear Caputo fractional Volterra integro-differential equations by power series and so on. Alaroud, M. et al. [80] and Alaroud, M. et al. [81] used FPS to solve fuzzy fractional differential equations.

Table 1
Comparison F of the numerical approaches to get solutions of the fuzzy fractional differential equations.

Year	Name of the authors	Type of problem	Techniques
2020	Allaviranloo et al. [72]	Fuzzy fractional differential equation	Fuzzy generalized Taylor theorem
2021	Khakrangin et al. [75]	Fuzzy fractional differential equation	Haar wavelet
2022	Haq et al. [76]	Fuzzy fractional differential equation	Reliable method
2024	In this article	Fuzzy fractional differential equation	Fuzzy Fractional Taylor's series under Caputo derivative under generalized Hukuhara difference

1.6. Taylor series exponential for differential equation

Background of the Taylor series exponential and its application in differential equations are discussed in this section. It is a sum of an infinite series of a derivative of functions on a particular point. The Taylor series exponential is used to evaluate the solutions of differential equations. Zhang, Z. et al. [82] used the Taylor series to solve the nonlinear differential equations and Jumarie, G. [83] applied the Taylor series to solve PDE and nondifferentiable equations. Abad, A. et al. [84] applied the Taylor series integrator to find out the solution to differential equations and Ren, Y. et al. [85] used the Taylor series to handle second kind differential equations.

1.7. Motivation and novelty of this study

The main purpose of this research is to solve the fuzzy factional differential equations using the fuzzy power series approximation, Taylor series exponential and fuzzy Caputo fractional derivative. Here, we solve the fractional order differential equation and check the convergence of the series solution. Lastly, examine the solution using numerical examples the draw the curve of the solution functions in 2D and 3D spaces. The result of our study is explored elaborately and future extensions are discussed in the conclusion.

1.8. Structure of this study

This research, contains an introduction to the differential equation, factional differential equation and fuzzy factional differential equation with their background in Section 1. In Section 2 discussed on preliminaries about fuzzy sets and its properties. The fuzzy fractional counterpart of Taylor's series approximation is introduced in Section 3. Section 4 contains the numerical illustration followed by result & discussion in Section 5. Finally, the conclusion of this study is discussed in Section 6.

2. Preliminaries

In this present section, some preliminaries on the fuzzy set and logic are recapitulated. The Fuzzy sets [26] are the set of collections of objects associated with membership values distinguished from crisp sets. Fuzzy numbers [86,87] are a special type of fuzzy set that satisfies some properties, which are discussed in detail in this section. Also, $\alpha-$cut of a fuzzy set is another representation that is also used in this study, discussed by Mukherjee et al. [86].

Definition 1 (*Fuzzy Set [88]*)**.** An ordered pair made up of a fuzzy set $\tilde{\eta}$ and its membership function $\mu_{\tilde{\eta}}(\chi)$ is known as a fuzzy set. A set is referred to as a discrete fuzzy set if it is created using a discrete universe and it is given as $\left\{\frac{\mu_{\tilde{\eta}}(\chi_1)}{\chi_1}, \frac{\mu_{\tilde{\eta}}(\chi_2)}{\chi_2}, \frac{\mu_{\tilde{\eta}}(\chi_3)}{\chi_3}, \dots\right\}$. Set is referred to as a continuous fuzzy set if it is created using a continuous universe. Another representation of fuzzy set $\tilde{\eta}$ is $\tilde{\eta} = \left\{\left(\chi_1, \mu_{\tilde{\eta}}(\chi_1)\right), \left(\chi_2, \mu_{\tilde{\eta}}(\chi_2)\right), \left(\chi_3, \mu_{\tilde{\eta}}(\chi_3)\right), \dots\right\}$.

Definition 2 (*Fuzzy Number [86]*)**.** Consider, $\mathbb{R}$ be a set of real numbers is a universal set of discourse. Then $\tilde{E}$ be a fuzzy set defined on $\mathbb{R}$ and $\tilde{E} = \{\tilde{\eta} : R \to [01]\}$ be the collection of fuzzy numbers, satisfying the following properties

(a) Membership value of $\tilde{\eta}$ (i.e., $\mu_{\tilde{\eta}}$) should be 1 for at least one point.
(b) $\alpha-$cut must be closed. The parametric form of ${}^{\alpha}\tilde{\eta} = \left[\underline{\eta}, \overline{\eta}\right]$ where $\underline{\eta} \leq \overline{\eta}$.
(c) Support must be bounded.
(d) The set $\tilde{E}$ is convex fuzzy set, i.e., $\mu_{\tilde{E}}\left(\lambda t_1 + (1-\lambda)t_2\right) \geq \min\left\{\mu_{\tilde{E}}\left(t_1\right), \mu_{\tilde{E}}\left(t_2\right)\right\}$ for all $t_1, t_2 \in \tilde{E}$ and $\lambda \in [0,1]$.

Definition 3 (*Triangular Fuzzy Number (TFN) [86]*)**.** The triangular fuzzy number $(\tilde{\eta})$ is denoted as triplet $\tilde{\eta} = (p, q, r)$ where $p, q, r \in \mathbb{R}$ with $p \leq q \leq r$ and its membership function $\left(\mu_{\tilde{\eta}}(\chi)\right)$ is given as

$$\mu_{\tilde{\eta}}(\chi) = \begin{cases} \dfrac{\chi - p}{q - p} \ ; \ p < \chi \leq q \\ \dfrac{r - \chi}{r - q} \ ; \ q < \chi \leq r \\ 0 \quad ; \ otherwise \end{cases} \tag{1}$$

The $\alpha-$cut of $\tilde{\eta}$ is ${}^{\alpha}\tilde{\eta} = [p + (q-p)\alpha, r - (r-q)\alpha]$ where $0 \leq \alpha \leq 1$.

Remarks 1. The geometric representation of Triangular Fuzzy Number (TFN) are presented in Fig. 1. The structural diagram of the membership function $\left(\mu_{\tilde{\eta}}\right)$ of TFN $\tilde{\eta} = (p, q, r)$ are presented. The $\alpha-$cut, support and core of fuzzy number are shown [86].

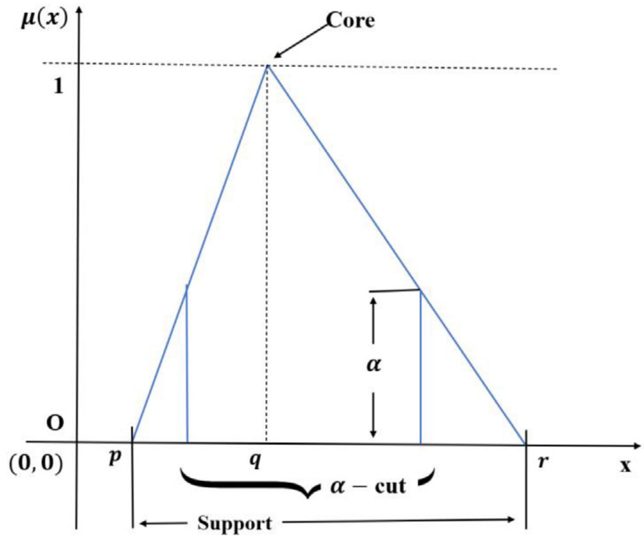

Fig. 1. Graphical representation of Triangular Fuzzy Number (TFN).

Definition 4 (*Arithmetic Operation of TFNs [86]*). Let $\tilde{\eta}, \tilde{\xi} \in E$ are triangular fuzzy numbers and k is scalar then the arithmetic operations between $\tilde{\eta}$ and $\tilde{\xi}$ for all $\alpha \in (01]$, are as follows

$$^{\alpha}\tilde{\eta} \oplus {}^{\alpha}\tilde{\xi} = \left[\underline{\eta}, \overline{\eta}\right] \oplus \left[\underline{\xi}, \overline{\xi}\right] = \left[\underline{\eta} + \underline{\xi}, \overline{\eta} + \overline{\xi}\right] \tag{2}$$

$$k \otimes {}^{\alpha}\tilde{\eta} = \begin{cases} \left[k\underline{\eta}, k\overline{\eta}\right] & k \geq 0 \\ \left[k\overline{\eta}, k\underline{\eta}\right] & k < 0 \end{cases} \tag{3}$$

where $^{\alpha}\tilde{\eta}$ and $^{\alpha}\tilde{\xi}$ are $\alpha-$cut of $\tilde{\eta}$ and $\tilde{\xi}$, respectively.

Definition 5 (*Multiplication of TFNs [86]*). Let $\tilde{\eta}, \tilde{\xi} \in E$ are triangular fuzzy numbers, then

$$^{\alpha}\tilde{\eta} \otimes {}^{\alpha}\tilde{\xi} = \left[\underline{\eta}, \overline{\eta}\right] \otimes \left[\underline{\xi}, \overline{\xi}\right] = \left[\min\left\{\underline{\eta}\underline{\xi}, \underline{\eta}\overline{\xi}, \overline{\eta}\underline{\xi}, \overline{\eta}\overline{\xi}\right\}, \max\left\{\underline{\eta}\underline{\xi}, \underline{\eta}\overline{\xi}, \overline{\eta}\underline{\xi}, \overline{\eta}\overline{\xi}\right\}\right] \tag{4}$$

If both $\tilde{\eta}, \tilde{\xi} \in E$ are positive fuzzy numbers, then

$$^{\alpha}\tilde{\eta} \otimes {}^{\alpha}\tilde{\xi} = \left[\underline{\eta}, \overline{\eta}\right] \otimes \left[\underline{\xi}, \overline{\xi}\right] = \left[\underline{\eta}\underline{\xi}, \overline{\eta}\overline{\xi}\right] \tag{5}$$

Definition 6 (*Generalized Hukuhara Difference [28,47]*). Consider $\tilde{\eta}$ and $\tilde{\xi}$ are two fuzzy numbers and $^{\alpha}\tilde{\eta} = \left[\underline{\eta}, \overline{\eta}\right]$ and $^{\alpha}\tilde{\xi} = \left[\underline{\xi}, \overline{\xi}\right]$ are the $\alpha-$cut of them, respectively. Then the fuzzy difference (i.e., Generalized Hukuhara difference) between $^{\alpha}\tilde{\eta}$ and $^{\alpha}\tilde{\xi}$ is given as

$$^{\alpha}\tilde{\eta} \ominus {}^{\alpha}\tilde{\xi} = \left[\min\left\{\underline{\eta} - \underline{\xi}, \overline{\eta} - \overline{\xi}\right\}, \max\left\{\underline{\eta} - \underline{\xi}, \overline{\eta} - \overline{\xi}\right\}\right] \tag{6}$$

The Hausdorff distance between fuzzy numbers [89] is given, $\mathcal{H} : E \times E \rightarrow R^{+} \cup 0$ as

$$\mathcal{H}\left({}^{\alpha}\tilde{\eta}, {}^{\alpha}\tilde{\xi}\right) = Sup \max_{0 \leq \alpha \leq 1} \left\{\left[\min\left\{\underline{\eta} - \underline{\xi}, \overline{\eta} - \overline{\xi}\right\}, \max\left\{\underline{\eta} - \underline{\xi}, \overline{\eta} - \overline{\xi}\right\}\right]\right\} \tag{7}$$

Theorem 1. *The metric space* $(E, \mathcal{H})$ *is complete, separable and locally compact [89–91] with following cases,*

(1) $\mathcal{H}\left({}^{\alpha}\tilde{\eta}, {}^{\alpha}\tilde{\xi}\right) = \mathcal{H}\left({}^{\alpha}\tilde{\xi}, {}^{\alpha}\tilde{\eta}\right); \forall \tilde{\eta}, \tilde{\xi} \in E$
(2) $\mathcal{H}\left(k\left({}^{\alpha}\tilde{\eta}, {}^{\alpha}\tilde{\xi}\right)\right) = |k| \mathcal{H}\left({}^{\alpha}\tilde{\eta}, {}^{\alpha}\tilde{\xi}\right); \forall k \in R$ and $\tilde{\eta}, \tilde{\xi} \in E$.
(3) $\mathcal{H}\left({}^{\alpha}\tilde{\eta} \oplus {}^{\alpha}\tilde{\xi}, {}^{\alpha}\tilde{\tau} \oplus {}^{\alpha}\tilde{\rho}\right) \leq \mathcal{H}\left({}^{\alpha}\tilde{\eta}, {}^{\alpha}\tilde{\tau}\right) \oplus \mathcal{H}\left({}^{\alpha}\tilde{\xi}, {}^{\alpha}\tilde{\rho}\right); \forall \tilde{\eta}, \tilde{\xi}, \tilde{\tau}, \tilde{\rho} \in E$.

Note 1: The properties of complete metric space are described by Avilés et al. [92], the separable metric spaces are described by Soileau, K. M. [93] and compact & locally compact spaces are discussed by Dranishnikov, A. N. [94]. In Theorem 1, these concepts are extended in fuzzy environments.

3. Fuzzy fractional counterpart of Taylor's series approximation

In this section, the fuzzy fractional differential equation is solved using fuzzy fractional Taylor's series. Allahviranloo, T. et al. [72] applied fuzzy generalized Taylor theorem to solve the fuzzy factional differential equation. Senol, M. et al. [95] use Taylor series convergence analysis for solving partial differential equations (PDE). Also, Alaroud, M. et al. [80] utilize the fuzzy Caputo fractional derivative under the Taylor series approximation to solve the fuzzy fractional differential equations. Here, the fuzzy fractional Taylor's expansion is done under fuzzy Caputo fractional derivative [74]. The convergence of the fuzzy power series containing fraction terms is proved. This section consists the subsections as follows:

3.1. Fractional derivative of fuzzy valued functions

The fuzzy fractional derivative of fuzzy functions is described by Harir, A. et al. [96]. The fractional derivative under the fuzzy Caputo is described by Allahviranloo, T. et al. [97] and Ain, Q. T. et al. [98]. The definitions and example of fractional derivative of fuzzy valued functions are discussed as follows:

Definition 7. A fuzzy valued function in parametric form [98] is defined as

$$^{\alpha}\tilde{g}\left(\tilde{\xi}\right) = \left[\min\left\{\underline{g}\left(\underline{\xi},\overline{\xi}\right),\overline{g}\left(\underline{\xi},\overline{\xi}\right)\right\},\max\left\{\underline{g}\left(\underline{\xi},\overline{\xi}\right),\overline{g}\left(\underline{\xi},\overline{\xi}\right)\right\}\right] \tag{8}$$

There are several types of fuzzy fractional derivatives, like Differentiable Type Fuzzy Fractional Derivative (DTFFD), Caputo-Type Fuzzy Fractional Derivative with Generalized Differentiability (CTFFDGD), Jumarie Type Fuzzy Fractional Derivative (JTFFD), Yang-Type Fuzzy Fractional Derivative (YTFFD), Atangana–Baleanu–Caputo Type Fuzzy Fractional Derivative (ABC-TFFD) and Caputo Type Fuzzy Fractional Derivative (CTFFD). This study considered the Caputo Type Fuzzy Fractional Derivative (CTFFD) [23]. The Caputo factional derivative [24] is more useful for this study based on initial condition, real world phenomena, physical interpretations, numerical simulations, smoother transitions, etc. The Caputo factional derivatives discussed as follows:

Definition 8. The Caputo fractional derivative [97] of order $\beta \in (0,1]$ of a fuzzy valued function $\tilde{g}\left(\tilde{\xi}\right)$ is denoted as $D^{\beta}\tilde{g}\left(\tilde{\xi}\right)$ and is given by

$$D^{\beta}\tilde{g}\left(\tilde{\xi}\right) = \frac{1}{\Gamma(n-\beta)}\int_{0}^{\tilde{\xi}}\left(\tilde{\xi}\ominus\tilde{\eta}\right)^{n-\beta-1}\tilde{g}^{n}(\tilde{\eta})\,d\tilde{\eta} \tag{9}$$

The parametric form or $\alpha-$cut of Eq. (9) produces the following derivation:

$$D^{\beta}\tilde{g}\left(\tilde{\xi}\right) = \frac{1}{\Gamma(n-\beta)}\int_{0}^{\tilde{\xi}}\left({}^{\alpha}\tilde{\xi}\ominus{}^{\alpha}\tilde{\eta}\right)^{n-\beta-1}\tilde{g}^{n}({}^{\alpha}\tilde{\eta})\,d\tilde{\eta} \tag{10}$$

That is

$$\left[\min D^{\beta}\left\{\underline{g}\left(\underline{\xi},\overline{\xi}\right),\overline{g}\left(\underline{\xi},\overline{\xi}\right)\right\},\max D^{\beta}\left\{\underline{g}\left(\underline{\xi},\overline{\xi}\right),\overline{g}\left(\underline{\xi},\overline{\xi}\right)\right\}\right]$$
$$= \left[\frac{1}{\Gamma(n-\beta)}\int_{0}^{\underline{\xi}}\min((\underline{\xi}-\underline{\eta})^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1})\underline{g}^{n}d\underline{\eta},\frac{1}{\Gamma(n-\beta)}\int_{0}^{\overline{\xi}}\max((\underline{\xi}-\underline{\eta})^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1})\overline{g}^{n}d\overline{\eta}\right] \tag{11}$$

Using the interval arithmetic, we get the following equations:

$$D^{\beta}{}_{CL}\underline{g} = \min D^{\beta}\left\{\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}\right\} = \frac{1}{\Gamma(n-\beta)}\int_{0}^{\underline{\xi}}\min\left\{\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}\right\}\underline{g}^{n}d\underline{\eta} \tag{12}$$

$$D^{\beta}{}_{CU}\overline{g} = \max D^{\beta}\left\{\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}\right\} = \frac{1}{\Gamma(n-\beta)}\int_{0}^{\overline{\xi}}\max\left\{\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1},\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}\right\}\overline{g}^{n}d\overline{\eta} \tag{13}$$

In Eqs. (12) and (13), $D^{\beta}{}_{CL}\underline{g}$, $D^{\beta}{}_{CU}\overline{g}$ are fuzzy fractional Caputo lower and upper $\alpha-cut$ derivatives respectively.

Example 1. Evaluate $D^{\beta}\left(\tilde{\xi}^{n}\right)$ at $\tilde{\xi}_0 = \tilde{1}$ when $\beta \in (0,1]$.

Solution: Taking $\alpha-$cut of the fuzzy fractional derivative $D^{\beta}\left(\tilde{\xi}^{n}\right)$ of the fuzzy valued function $\tilde{\xi}^{n}$, we get

$$D^{\beta}\left({}^{\alpha}\tilde{\xi}^{n}\right) = \left[D^{\beta}\underline{\xi}^{n}, D^{\beta}\overline{\xi}^{n}\right]$$

And, the $\alpha-$cut representation of the given condition is

$$^{\alpha}\tilde{\xi}_0 = {}^{\alpha}\tilde{1} = \left[\underline{\xi}_0,\overline{\xi}_0\right] = [0.5+0.5\alpha, 1.5-0.5\alpha]$$

Following the discussion as the consequence of Definition 8, we get

$$\left[D^{\beta}\left(\underline{\xi}^{n}\right),D^{\beta}\left(\overline{\xi}^{n}\right)\right]$$
$$= \left[\min\left\{\frac{n!}{\Gamma(n-\beta)}\int_{0}^{\underline{\xi}}\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1}d\underline{\eta},\frac{n!}{\Gamma(n-\beta)}\int_{0}^{\overline{\xi}}\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}d\overline{\eta}\right\},\right.$$
$$\left.\max\left\{\frac{n!}{\Gamma(n-\beta)}\int_{0}^{\underline{\xi}}\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta-1}d\underline{\eta},\frac{n!}{\Gamma(n-\beta)}\int_{0}^{\overline{\xi}}\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta-1}d\overline{\eta}\right\}\right]$$

Here, fuzzy initial condition is positive, so above expression becomes as follows,

$$\left[D^{\beta}\left(\underline{\xi}^{n}\right),D^{\beta}\left(\overline{\xi}^{n}\right)\right] = \left[\frac{n!}{\Gamma(n-\beta)}\left(\frac{\left(\underline{\xi}-\underline{\eta}\right)^{n-\beta}}{n-\beta}\right)_{0}^{\underline{\xi}},\frac{n!}{\Gamma(n-\beta)}\left(\frac{\left(\overline{\xi}-\overline{\eta}\right)^{n-\beta}}{n-\beta}\right)_{0}^{\overline{\xi}}\right]$$

$$= \left[\frac{n!}{\Gamma(n-\beta)} \frac{\left(\underline{\xi}\right)^{n-\beta}}{n-\beta}, \frac{n!}{\Gamma(n-\beta)} \frac{\left(\overline{\xi}\right)^{n-\beta}}{n-\beta} \right]$$

$$= \left[\frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\underline{\xi}\right)^{n-\beta}, \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\overline{\xi}\right)^{n-\beta} \right]$$

At $\underline{\xi}_0 = 0.5 + 0.5\alpha$, the value of $D^\beta \left(\underline{\xi}_0^n\right)$ evaluated as

$$D^\beta \left(\underline{\xi}_0^n\right) = \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} (0.5+0.5\alpha)^{n-\beta}$$

and at $\overline{\xi}_0 = 1.5 - 0.5\alpha$, the value of $D^\beta \left(\overline{\xi}_0^n\right)$ evaluated as

$$D^\beta \left(\overline{\xi}_0^n\right) = \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} (1.5-0.5\alpha)^{n-\beta}$$

Thus, the fuzzy fractional derivative of $\tilde{\xi}^n$ in parametric form of at any arbitrary point,

$$D^\beta \left({}^\alpha\tilde{\xi}^n\right) = \left[\min\left\{ \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\underline{\xi}\right)^{n-\beta}, \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\overline{\xi}\right)^{n-\beta} \right\}, \max\left\{ \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\underline{\xi}\right)^{n-\beta}, \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} \left(\overline{\xi}\right)^{n-\beta} \right\}\right] \tag{14}$$

For validation of result, we put $\alpha = 1$ and $\beta = 1$ both values of $D^\beta \left(\underline{\xi}^n\right)$ and $D^\beta \left(\overline{\xi}^n\right)$ coincide.

3.2. Fuzzy power series

In this section, we discussed about fuzzy power series and properties of this series. The fuzzy power series defined, various properties of power series converge are described by Newman, W. G. [99]. And fuzzy power series applied in fuzzy differential equation to evaluate the solution. Shiri et al. [100] solve the fuzzy factional differential equation using fuzzy power series. To solve the fuzzy Legendre differential equation, use the fuzzy power series applied by Sabzi et al. [101]. The fuzzy power series and its properties are described as follows:

We consider a series,

$$\begin{aligned} \tilde{\eta}\left(\tilde{\xi}\right) &= \sum_{n=0}^{\infty} \tilde{a}_n \otimes \left(\tilde{\xi} \ominus \tilde{\xi}_0\right)^n \\ &= \tilde{a}_n \oplus \left\{ \tilde{a}_1 \otimes \left(\tilde{\xi} \ominus \tilde{\xi}_0\right)^1 \right\} \oplus \left\{ \tilde{a}_2 \otimes \left(\tilde{\xi} \ominus \tilde{\xi}_0\right)^2 \right\} \oplus \left\{ \tilde{a}_3 \otimes \left(\tilde{\xi} \ominus \tilde{\xi}_0\right)^3 \right\} \oplus \cdots \end{aligned} \tag{15}$$

Eq. (15) represent a fuzzy version of the power series about the fuzzy number $\tilde{\xi}_0$, where $\tilde{a}_n$s are fuzzy coefficients of the power series. Taking $\alpha-$cut on both sides of the above equation, we get

$${}^\alpha\tilde{\eta}\left(\tilde{\xi}\right) = {}^\alpha\tilde{a}_0 \oplus \left\{ {}^\alpha\tilde{a}_1 \otimes \left({}^\alpha\tilde{\xi} \ominus {}^\alpha\tilde{\xi}_0\right)^1 \right\} \oplus \left\{ {}^\alpha\tilde{a}_2 \otimes \left({}^\alpha\tilde{\xi} \ominus {}^\alpha\tilde{\xi}_0\right)^2 \right\} \oplus \left\{ {}^\alpha\tilde{a}_3 \otimes \left({}^\alpha\tilde{\xi} \ominus {}^\alpha\tilde{\xi}_0\right)^3 \right\} \oplus \cdots \tag{16}$$

The parametric counterpart of the above level cut representation is given as follows:

$$\begin{aligned} \left[\underline{\eta}\left(\underline{\xi}, \overline{\xi}\right), \overline{\eta}\left(\underline{\xi}, \overline{\xi}\right)\right] &= \left[\min\left\{\underline{a}_0, \overline{a}_0\right\}, \max\left\{\underline{a}_0, \overline{a}_0\right\}\right] \\ &\oplus \left\{ \left[\min\left\{\underline{a}_1, \overline{a}_1\right\}, \max\left\{\underline{a}_1, \overline{a}_1\right\}\right] \otimes \left[\min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^1, \left(\overline{\xi} - \overline{\xi}_0\right)^1 \right\}, \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^1, \left(\overline{\xi} - \overline{\xi}_0\right)^1 \right\}\right] \right\} \\ &\oplus \left\{ \left[\min\left\{\underline{a}_2, \overline{a}_2\right\}, \max\left\{\underline{a}_2, \overline{a}_2\right\}\right] \otimes \left[\min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^2, \left(\overline{\xi} - \overline{\xi}_0\right)^2 \right\}, \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^2, \left(\overline{\xi} - \overline{\xi}_0\right)^2 \right\}\right] \right\} \\ &\oplus \left\{ \left[\min\left\{\underline{a}_3, \overline{a}_3\right\}, \max\left\{\underline{a}_3, \overline{a}_3\right\}\right] \right. \\ &\left. \otimes \left[\min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^3, \left(\overline{\xi} - \overline{\xi}_0\right)^3 \right\}, \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^3, \left(\overline{\xi} - \overline{\xi}_0\right)^3 \right\}\right] \right\} \\ &\oplus \cdots \end{aligned} \tag{17}$$

Eq. (17) can be put in standard form as follows:

$$\begin{aligned} &\left[\underline{\eta}\left(\underline{\xi}, \overline{\xi}\right), \overline{\eta}\left(\underline{\xi}, \overline{\xi}\right)\right] \\ &= \sum_{n=0} \left\{ \left[\min\left\{\underline{a}_n, \overline{a}_n\right\}, \max\left\{\underline{a}_n, \overline{a}_n\right\}\right] \otimes \left[\min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}, \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}\right] \right\} \end{aligned} \tag{18}$$

That is,

$$\begin{aligned} &\left[\underline{\eta}\left(\underline{\xi}, \overline{\xi}\right), \overline{\eta}\left(\underline{\xi}, \overline{\xi}\right)\right] \\ &= \sum_{n=0} \left[\min\left\{ \min\left\{\underline{a}_n, \overline{a}_n\right\} \min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}, \right.\right. \\ &\min\left\{\underline{a}_n, \overline{a}_n\right\} \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}, \\ &\max\left\{\underline{a}_n, \overline{a}_n\right\} \min\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}, \\ &\max\left\{\underline{a}_n, \overline{a}_n\right\} \max\left\{ \left(\underline{\xi} - \underline{\xi}_0\right)^n, \left(\overline{\xi} - \overline{\xi}_0\right)^n \right\}\right\}, \end{aligned}$$

$$\max\left\{\min\left\{\underline{a}_n,\overline{a}_n\right\}\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\},\right.$$
$$\min\left\{\underline{a}_n,\overline{a}_n\right\}\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\},$$
$$\max\left\{\underline{a}_n,\overline{a}_n\right\}\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\},$$
$$\left.\max\left\{\underline{a}_n,\overline{a}_n\right\}\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\}\right\}\Bigg] \tag{19}$$

For sake of convenience, we are using following notations

$\underline{b}_n=\min\left\{\underline{a}_n,\overline{a}_n\right\}$, $\overline{b}_n=\max\left\{\underline{a}_n,\overline{a}_n\right\}$, $\left(\underline{\eta}-\underline{\eta}_0\right)^n=\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\}$, and
$\left(\overline{\eta}-\overline{\eta}_0\right)^n=\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\overline{\xi}-\overline{\xi}_0\right)^n\right\}$.

Thus, Eq. (19) is converted into the below mentioned equation:

$$\left[\underline{\eta}\left(\underline{\xi},\overline{\xi}\right),\overline{\eta}\left(\underline{\xi},\overline{\xi}\right)\right]=\sum_{n=0}\left[\underline{b}_n\left(\underline{\eta}-\underline{\eta}_0\right)^n,\overline{b}_n\left(\overline{\eta}-\overline{\eta}_0\right)^n\right] \tag{20}$$

When the fuzzy initial condition is positive, then we can use the notations, $\underline{\eta}\left(\underline{\xi},\overline{\xi}\right)=\underline{\eta}\left(\underline{\xi}\right)$, $\overline{\eta}\left(\underline{\xi},\overline{\xi}\right)=\overline{\eta}\left(\overline{\xi}\right)$ and thus Eq. (20) will be converted as follows:

$$\left[\underline{\eta}\left(\underline{\xi}\right),\overline{\eta}\left(\overline{\xi}\right)\right]=$$
$$\left[\min\left\{\underline{a}_0+\underline{a}_1\left(\underline{\xi}-\underline{\xi}_0\right)^1+\underline{a}_2\left(\underline{\xi}-\underline{\xi}_0\right)^2+\cdots,\overline{a}_0+\overline{a}_1\left(\overline{\xi}-\overline{\xi}_0\right)^1+\overline{a}_2\left(\overline{\xi}-\overline{\xi}_0\right)^2+\cdots\right\},\right.$$
$$\left.\max\left\{\underline{a}_0+\underline{a}_1\left(\underline{\xi}-\underline{\xi}_0\right)^1+\underline{a}_2\left(\underline{\xi}-\underline{\xi}_0\right)^2+\cdots,\overline{a}_0+\overline{a}_1\left(\overline{\xi}-\overline{\xi}_0\right)^1+\overline{a}_2\left(\overline{\xi}-\overline{\xi}_0\right)^2+\cdots\right\}\right] \tag{21}$$

Note 2: Fuzzy power series is convergent if lower bound and upper bound of the series are convergent simultaneously. The convergent of the fuzzy power series describe and prove by Newman, W. G. [99].

3.3. Convergence of fuzzy power series in parametric form

This section is dealing with the convergency criteria of the fuzzy power series mentioned in the earlier subsection. Bataineh et al. [102] applied fuzzy power series in parametric form to check converges of the series and solve the initial value problem.

Theorem 2. *A series in Eq. (19) using Eq. (20) converges for any arbitrary point, when* $\left|\underline{\eta}-\underline{\eta}_0\right|<1$ *and* $\left|\overline{\eta}-\overline{\eta}_0\right|<1$.

Proof. We consider Eq. (20) of the previous subsection that is

$$\left[\underline{\eta}\left(\underline{\xi},\overline{\xi}\right),\overline{\eta}\left(\underline{\xi},\overline{\xi}\right)\right]=\left[\underline{b}_n\left(\underline{\eta}-\underline{\eta}_0\right)^n,\overline{b}_n\left(\overline{\eta}-\overline{\eta}_0\right)^n\right]$$

Applying root test and let $\underline{l}=\lim_{n\longrightarrow\infty}\left|\underline{b}_n^{\frac{1}{n}}\right|$, $\overline{l}=\lim_{n\longrightarrow\infty}\left|\overline{b}_n^{\frac{1}{n}}\right|$ then Eq. (21) converges, when $\underline{l}\left|\underline{\eta}-\underline{\eta}_0\right|<1$ and $\overline{l}\left|\overline{\eta}-\overline{\eta}_0\right|<1$, which gives $\left|\underline{\eta}-\underline{\eta}_0\right|<\frac{1}{\lim_{n\to\infty}\left(\underline{b}_n\right)^{\frac{1}{n}}}$, $\left|\overline{\eta}-\overline{\eta}_0\right|<\frac{1}{\lim_{n\to\infty}\left(\overline{a}_n\right)^{\frac{1}{n}}}$. This proves the convergence of fuzzy power series.

3.4. Fuzzy fractional power series in parametric form due to Taylor's expansion

This section is dealing with the fuzzy fractional power series approximation due to Taylor's expansion is introduced and proved. Sala et al. [103] applied Taylor series on nonlinear fuzzy power series model and Yang et al. [77] use Taylor series in fuzzy model for Robots.

Theorem 3. *If a fuzzy function* $\tilde{g}:E\to E$ *is* n *times differentiable at* $\tilde{x}_0$ *in sense of Caputo fractional derivative, then its fractional Taylor's series in parametric form is given as follows:*

$$\left[\underline{\eta}\left(\underline{\xi},\overline{\xi}\right),\overline{\eta}\left(\underline{\xi},\overline{\xi}\right)\right]=\left[\min\left\{\underline{\eta}\left(\underline{\xi}_0\right),\overline{\eta}\left(\overline{\xi}_0\right)\right\},\max\left\{\underline{\eta}\left(\underline{\xi}_0\right),\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right]$$
$$\oplus\left\{\frac{\Gamma(2-\beta)}{\Gamma(2)}\left[\min\left\{D^{\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\},\max\left\{D^{\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right]\right.$$
$$\left.\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\overline{\xi}-\overline{\xi}_0\right)^1\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\overline{\xi}-\overline{\xi}_0\right)^1\right\}\right]\right\}$$
$$\oplus\left\{\frac{\Gamma(3-\beta)}{\Gamma(3)}\left[\min\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\},\max\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right]\right.$$
$$\left.\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\overline{\xi}-\overline{\xi}_0\right)^2\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\overline{\xi}-\overline{\xi}_0\right)^2\right\}\right]\right\}$$
$$\oplus\left\{\frac{\Gamma(4-\beta)}{\Gamma(4)}\left[\min\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\},\max\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right]\right.$$
$$\left.\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\overline{\xi}-\overline{\xi}_0\right)^3\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\overline{\xi}-\overline{\xi}_0\right)^3\right\}\right]\right\}$$
$$\oplus\cdots$$

Proof. We consider a series in parametric form as in Eq. (18),

$$\left[\underline{\eta}\left(\underline{\xi},\bar{\xi}\right),\bar{\eta}\left(\underline{\xi},\bar{\xi}\right)\right]=\sum_{n=0}\left[\min\left\{\underline{a}_n,\bar{a}_n\right\},\max\left\{\underline{a}_n,\bar{a}_n\right\}\right]\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\bar{\xi}-\bar{\xi}_0\right)^n\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^n,\left(\bar{\xi}-\bar{\xi}_0\right)^n\right\}\right]$$

For finding the coefficient $\underline{a}_0$, $\bar{a}_0$, $\underline{a}_1$ and $\bar{a}_1$, we differentiate above equation in the sense of fuzzy fractional Caputo derivative of order β and put $\underline{\xi}=\underline{\xi}_0$. Then, we have the following result:

$$\begin{aligned}\left[D^\beta\underline{\eta}\left(\underline{\xi},\bar{\xi}\right),D^\beta\bar{\eta}\left(\underline{\xi},\bar{\xi}\right)\right]=&\ D^\beta\left[\min\left\{\underline{a}_0,\bar{a}_0\right\},\max\left\{\underline{a}_0,\bar{a}_0\right\}\right]\\&\oplus\left\{D^\beta\left[\min\left\{\underline{a}_1,\bar{a}_1\right\},\max\left\{\underline{a}_1,\bar{a}_1\right\}\right]\right.\\&\left.\otimes D^\beta\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\bar{\xi}-\bar{\xi}_0\right)^1\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\bar{\xi}-\bar{\xi}_0\right)^1\right\}\right]\right\}\\&\oplus\left\{D^\beta\left[\min\left\{\underline{a}_2,\bar{a}_2\right\},\max\left\{\underline{a}_2,\bar{a}_2\right\}\right]\right.\\&\left.\otimes D^\beta\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\bar{\xi}-\bar{\xi}_0\right)^2\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\bar{\xi}-\bar{\xi}_0\right)^2\right\}\right]\right\}\\&\oplus\left\{D^\beta\left[\min\left\{\underline{a}_3,\bar{a}_3\right\},\max\left\{\underline{a}_3,\bar{a}_3\right\}\right]\right.\\&\left.\otimes D^\beta\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\bar{\xi}-\bar{\xi}_0\right)^3\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\bar{\xi}-\bar{\xi}_0\right)^3\right\}\right]\right\}\\&\oplus\cdots\end{aligned}$$

Since, the value of $n-1<\beta<n$. So, whenever the value goes greater than n, we consider that term is 0, which gives,

$$\left[\underline{a}_0,\bar{a}_0\right]=\left[\min\left\{\underline{\eta}\left(\underline{\xi}_0\right),\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{\underline{\eta}\left(\underline{\xi}_0\right),\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]$$

$$\left[\underline{a}_1,\bar{a}_1\right]=\frac{\Gamma(2-\beta)}{\Gamma(2)}\left[\min\left\{D^\beta\underline{\eta}\left(\underline{\xi}_0\right),D^\beta\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^\beta\underline{\eta}\left(\underline{\xi}_0\right),D^\beta\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]$$

We differentiate Eq. (17) in expanded form of order 2β, and put $\underline{x}=\underline{x}_0$, which produces

$$\left[\underline{a}_2,\bar{a}_2\right]=\frac{\Gamma(3-\beta)}{\Gamma(3)}\left[\min\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]$$

Continuing this process,

$$\begin{aligned}\left[\underline{a}_3,\bar{a}_3\right]&=\frac{\Gamma(4-\beta)}{\Gamma(4)}\left[\min\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\\ \vdots&=\ \vdots\\ \left[\underline{a}_n,\bar{a}_n\right]&=\frac{\Gamma(n+1-\beta)}{\Gamma(n+1)}\left[\min\left\{D^{n\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{n\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^{n\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{n\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\end{aligned}$$

Putting all values in Eq. (18),

$$\begin{aligned}\left[\underline{\eta}\left(\underline{\xi},\bar{\xi}\right),\bar{\eta}\left(\underline{\xi},\bar{\xi}\right)\right]=&\left[\min\left\{\underline{\eta}\left(\underline{\xi}_0\right),\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{\underline{\eta}\left(\underline{\xi}_0\right),\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\\&\oplus\frac{\Gamma(2-\beta)}{\Gamma(2)}\left[\min\left\{D^\beta\underline{\eta}\left(\underline{\xi}_0\right),D^\beta\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^\beta\underline{\eta}\left(\underline{\xi}_0\right),D^\beta\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\\&\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\bar{\xi}-\bar{\xi}_0\right)^1\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1,\left(\bar{\xi}-\bar{\xi}_0\right)^1\right\}\right]\\&\oplus\frac{\Gamma(3-\beta)}{\Gamma(3)}\left[\min\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{2\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\\&\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\bar{\xi}-\bar{\xi}_0\right)^2\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2,\left(\bar{\xi}-\bar{\xi}_0\right)^2\right\}\right]\\&\oplus\frac{\Gamma(4-\beta)}{\Gamma(4)}\left[\min\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\},\max\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right),D^{3\beta}\bar{\eta}\left(\bar{\xi}_0\right)\right\}\right]\\&\otimes\left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\bar{\xi}-\bar{\xi}_0\right)^3\right\},\max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3,\left(\bar{\xi}-\bar{\xi}_0\right)^3\right\}\right]\\&\oplus\cdots\end{aligned}$$

This is the required Fuzzy Fractional Taylor's expansion.

3.5. Main problem

This section explains solution of the linear fuzzy fractional differential equation applying Fuzzy Fractional Taylor's theorem. Here, we consider a fuzzy factional differential equation and solve it using the above mentioned fuzzy fractional Taylor's expansion and Caputo fractional derivative.

The linear fuzzy fractional differential equation as in below Eq. (22) is used to model growth or decay equations in more realistic way. This Eq. (22) can also be considered to model fuzzy fractional inventory scenario and many other uses.

Consider a linear fuzzy fractional differential equation in general,

$$\frac{d^\beta\tilde{\eta}}{d\tilde{\xi}^\beta}=\tilde{A}\otimes\tilde{\eta}\oplus\tilde{B};\tilde{\eta}\left(\tilde{\xi}_0\right)=\tilde{\eta}_0 \tag{22}$$

where, $\tilde{A}=\begin{bmatrix}\tilde{a}_{11}&\cdots&\tilde{a}_{1n}\\\vdots&\ddots&\vdots\\\tilde{a}_{n1}&\cdots&\tilde{a}_{nn}\end{bmatrix}$ is $n\times n$ constant fuzzy matrix and $\tilde{B}=\begin{bmatrix}\tilde{b}_1\ or\ \tilde{b}_1(\tilde{\eta})\\\vdots\\\tilde{b}_n\ or\ \tilde{b}_n(\tilde{\eta})\end{bmatrix}$ is $n\times 1$ either constant or function of $\tilde{\eta}$.

To solve Eq. (22) by taking $\alpha-$cut on both sides of this equation, get

$$\frac{{}^{\alpha}d^{\beta}\tilde{\eta}}{d\tilde{\xi}^{\beta}} = {}^{\alpha}\tilde{A} \otimes {}^{\alpha}\tilde{\eta} \oplus {}^{\alpha}\tilde{B}; {}^{\alpha}\tilde{\eta}\left({}^{\alpha}\tilde{\xi}_0\right) = {}^{\alpha}\tilde{\eta}_0 \tag{23}$$

Writing parametric form of Eq. (23), using these notations, ${}^{\alpha}\tilde{A} = \left[\underline{A}, \overline{A}\right]$, ${}^{\alpha}\tilde{\eta} = \left[\underline{\eta}, \overline{\eta}\right]$, ${}^{\alpha}\tilde{B} = \left[\underline{B}, \overline{B}\right]$, ${}^{\alpha}\tilde{\eta}_0 = \left[\underline{\eta}_0, \overline{\eta}_0\right]$ and ${}^{\alpha}\tilde{\xi}_0 = \left[\underline{\xi}_0, \overline{\xi}_0\right]$.

Now, this Eq. (23) gets converted,

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = \left[\underline{A}, \overline{A}\right] \otimes \left[\underline{\eta}, \overline{\eta}\right] \oplus \left[\underline{B}, \overline{B}\right] \tag{24}$$

Which gives,

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = \left[\min\left\{\underline{A}\underline{\eta}, \underline{A}\overline{\eta}, \overline{A}\underline{\eta}, \overline{A}\overline{\eta}\right\}, \max\left\{\underline{A}\underline{\eta}, \underline{A}\overline{\eta}, \overline{A}\underline{\eta}, \overline{A}\overline{\eta}\right\}\right] \oplus \left[\underline{B}, \overline{B}\right]$$

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = \left[\min\left\{\underline{A}\underline{\eta}, \underline{A}\overline{\eta}, \overline{A}\underline{\eta}, \overline{A}\overline{\eta}\right\} + \underline{B}, \max\left\{\underline{A}\underline{\eta}, \underline{A}\overline{\eta}, \overline{A}\underline{\eta}, \overline{A}\overline{\eta}\right\} + \overline{B}\right] \tag{25}$$

Now, we can apply fuzzy fractional Taylor's theorem on Eq. (25).

First, coefficient value in parametric form as follows,

$$\left[\underline{a}_0, \overline{a}_0\right] = \left[\min\left\{\underline{A}\underline{\eta}\left(\underline{\xi}_0\right), \underline{A}\overline{\eta}\left(\underline{\xi}_0\right), \overline{A}\underline{\eta}\left(\overline{\xi}_0\right), \overline{A}\overline{\eta}\left(\overline{\xi}_0\right)\right\}, \max\left\{\underline{A}\underline{\eta}\left(\underline{\xi}_0\right), \underline{A}\overline{\eta}\left(\underline{\xi}_0\right), \overline{A}\underline{\eta}\left(\overline{\xi}_0\right), \overline{A}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \oplus \left[\underline{B}, \overline{B}\right]$$

$$\left[\underline{a}_1, \overline{a}_1\right] = \frac{\Gamma(2-\beta)}{\Gamma(2)}\left[\min\left\{D^{\beta}\underline{A}\underline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\underline{A}\overline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\overline{A}\underline{\eta}\left(\overline{\xi}_0\right), D^{\beta}\overline{A}\overline{\eta}\left(\overline{\xi}_0\right)\right\},\right.$$
$$\left.\max\left\{D^{\beta}\underline{A}\underline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\underline{A}\overline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\overline{A}\underline{\eta}\left(\overline{\xi}_0\right), D^{\beta}\overline{A}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \oplus \left[D^{\beta}\underline{B}, D^{\beta}\overline{B}\right] \cdots$$

Now, put all values in Fuzzy Taylor's theorem and obtain fuzzy solution. In next section, a numerical illustration is solved, based on proposed technique.

4. Numerical illustration

This section discusses the introduced theory with numerical examples with its Tabular and graphical outputs. Here consider two numerical examples of fuzzy factional differential equations and evaluate their solution spaces. Also, analysis the numerical solution spaces with the pictorial diagram in 2D and 3D spaces. The numerical examples are solved as follows:

Example 2. For doing such, we consider a fuzzy fractional differential equation as follows:

$$\frac{d^{\beta}\tilde{\eta}}{d\tilde{\xi}^{\beta}} = \tilde{2} \otimes \tilde{\eta} \oplus \tilde{3} \otimes \tilde{\xi} \tag{26}$$

Furthermore, the initial condition is given by $\tilde{\eta}\left(\tilde{0}\right) = \tilde{0}$. We want to calculate $\tilde{\eta}\left(\widetilde{0.2}\right)$.

Solution: For solving, we take $\alpha-$cut on both sides of Eq. (26),

$$\frac{{}^{\alpha}d^{\beta}\tilde{\eta}}{d\tilde{\xi}^{\beta}} = {}^{\alpha}\tilde{2} \otimes {}^{\alpha}\tilde{\eta} \oplus {}^{\alpha}\tilde{3} \otimes {}^{\alpha}\tilde{\xi} \tag{27}$$

with $\tilde{\xi}\left(\tilde{0}\right) = \tilde{0}$. The parametric form of Eq. (26) with initial condition provides, the following ${}^{\alpha}\tilde{2} = [1+\alpha, 3-\alpha]$, ${}^{\alpha}\tilde{3} = [2+\alpha, 4-\alpha]$, ${}^{\alpha}\tilde{0} = [-1+\alpha, 1-\alpha]$ and ${}^{\alpha}\widetilde{0.2} = [0.1+0.1\alpha, 0.3-0.1\alpha]$. Therefore, Eq. (27) will be of the form:

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = [1+\alpha, 3-\alpha] \otimes \left[\underline{\eta}, \overline{\eta}\right] \oplus [2+\alpha, 4-\alpha] \otimes \left[\underline{\xi}, \overline{\xi}\right] \tag{28}$$

Putting $\underline{\xi}_0 = (\alpha-1), \overline{\xi}_0 = (1-\alpha), \underline{\eta}_0 = (\alpha-1), \overline{\eta}_0 = (1-\alpha)$ in Eq. (28), we have

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = [1+\alpha, 3-\alpha] \otimes [(\alpha-1), (1-\alpha)] \oplus [2+\alpha, 4-\alpha] \otimes [(\alpha-1), (1-\alpha)]$$

That is, $\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = \left[-\alpha^2+4\alpha-3, \alpha^2-4\alpha+3\right] \oplus \left[-\alpha^2+5\alpha-4, \alpha^2-5\alpha+4,\right]$

That is, $\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = \left[-2\alpha^2+9\alpha-7, 2\alpha^2-9\alpha+7\right]$

Now, differentiating Eq. (27) 2β times, we get

$$\left[\frac{d\underline{\eta}^{2\beta}}{d\underline{\xi}^{2\beta}}, \frac{d\overline{\eta}^{2\beta}}{d\overline{\xi}^{2\beta}}\right] = \left([1+\alpha, 3-\alpha] \otimes \left[-2\alpha^2+9\alpha-7, 2\alpha^2-9\alpha+7\right]\right)$$
$$\oplus [2+\alpha, 4-\alpha] \otimes \left[(2+\alpha)\frac{1}{\Gamma(2-\beta)}(\alpha-1)^{1-\beta}, (4-\alpha)\frac{1}{\Gamma(2-\beta)}(1-\alpha)^{1-\beta}\right]$$

Table 2
The value of lower α−cut of $\underline{\eta}$ for different values of α and β.

α/β	0	0.2	0.4	0.6	0.8	1
0	3.90000000000000	1.94560000000000	0.664800000000001	0.0287999999999992	0.221599999999999	$8.88178419700125e^{-16}$
0.2	3.56378047780319	1.75720728160335	0.578014193535812	0.0542703442121331	0.220117889453173	$8.27234965843575e^{-16}$
0.4	3.37822521150968	1.65323574300428	0.530118213779071	0.0683271023444086	0.219299931544614	$7.93601050908146e^{-16}$
0.6	3.34759270576507	1.63607153733644	0.522211276377890	0.0706476709428577	0.219164898458066	$7.88048575286982e^{-16}$
0.8	3.49902683775883	1.72092409913278	0.561299825387218	0.0591757628212081	0.219832444835834	$8.15497662642671e^{-16}$
1	3.90000000000000	1.94560000000000	0.664800000000001	0.0287999999999992	0.221599999999999	$8.88178419700125e^{-16}$

Table 3
The value of upper α−cut of $\overline{\eta}$ for different values of α and β.

α/β	0	0.2	0.4	0.6	0.8	1
0	8.70000000000000	5.65760000000000	3.35280000000000	1.69920000000000	0.610400000000000	$8.88178419700125e^{-16}$
0.2	8.17165503654787	5.32428980591363	3.16391324475441	1.61005379525753	0.582239899610292	$8.27234965843575\ e^{-16}$
0.4	7.88006818951522	5.14034016069988	3.05966905351915	1.56085514179457	0.566698699347668	$7.93601050908146e^{-16}$
0.6	7.83193139477368	5.10997271990294	3.04245983682247	1.55273315170000	0.564133070703262	$7.88048575286982e^{-16}$
0.8	8.06989931647816	5.26009648308108	3.12753491407806	1.59288483012577	0.576816451880862	$8.15497662642671e^{-16}$
1	8.70000000000000	5.65760000000000	3.35280000000000	1.69920000000000	0.610400000000000	$8.88178419700125e^{-16}$

That is,

$$\left[\frac{d\underline{\eta}^{2\beta}}{d\underline{\xi}^{2\beta}}, \frac{d\overline{\eta}^{2\beta}}{d\overline{\xi}^{2\beta}}\right] = \left[2\alpha^3 - 15\alpha^2 + 34\alpha - 21, -2\alpha^3 + 15\alpha^2 - 34\alpha + 21\right] \oplus \left[(2+\alpha)^2 \frac{1}{\Gamma(2-\beta)} (\alpha-1)^{1-\beta}, (4-\alpha)^2 \frac{1}{\Gamma(2-\beta)} (1-\alpha)^{1-\beta}\right]$$

That is,

$$\left[\frac{d\underline{\eta}^{2\beta}}{d\underline{\xi}^{2\beta}}, \frac{d\overline{\eta}^{2\beta}}{d\overline{\xi}^{2\beta}}\right] = \left[2\alpha^3 - 15\alpha^2 + 34\alpha - 21 + (2+\alpha)^2 \frac{1}{\Gamma(2-\beta)} (\alpha-1)^{1-\beta}, -2\alpha^3 + 15\alpha^2 - 34\alpha + 21 + (4-\alpha)^2 \frac{1}{\Gamma(2-\beta)} (1-\alpha)^{1-\beta}\right]$$

Now we put these fractional derivatives $\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right], \left[\frac{d\underline{\eta}^{2\beta}}{d\underline{\xi}^{2\beta}}, \frac{d\overline{\eta}^{2\beta}}{d\overline{\xi}^{2\beta}}\right]$ of different order in fuzzy fractional Taylor's theorem in the earlier section.

$$\begin{aligned}\left[\underline{\eta}\left(\underline{\xi},\overline{\xi}\right), \overline{\eta}\left(\underline{\xi},\overline{\xi}\right)\right] &= \left[\min\left\{\underline{\eta}\left(\underline{\xi}_0\right), \overline{\eta}\left(\overline{\xi}_0\right)\right\}, \max\left\{\underline{\eta}\left(\underline{\xi}_0\right), \overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \\ &\oplus \frac{\Gamma(2-\beta)}{\Gamma(2)} \left[\min\left\{D^{\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}, \max\left\{D^{\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \\ &\otimes \left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1, \left(\overline{\xi}-\overline{\xi}_0\right)^1\right\}, \max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^1, \left(\overline{\xi}-\overline{\xi}_0\right)^1\right\}\right] \\ &\oplus \frac{\Gamma(3-\beta)}{\Gamma(3)} \left[\min\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{2\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}, \max\left\{D^{2\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{2\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \\ &\otimes \left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2, \left(\overline{\xi}-\overline{\xi}_0\right)^2\right\}, \max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^2, \left(\overline{\xi}-\overline{\xi}_0\right)^2\right\}\right] \\ &\oplus \frac{\Gamma(4-\beta)}{\Gamma(4)} \left[\min\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{3\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}, \max\left\{D^{3\beta}\underline{\eta}\left(\underline{\xi}_0\right), D^{3\beta}\overline{\eta}\left(\overline{\xi}_0\right)\right\}\right] \\ &\otimes \left[\min\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3, \left(\overline{\xi}-\overline{\xi}_0\right)^3\right\}, \max\left\{\left(\underline{\xi}-\underline{\xi}_0\right)^3, \left(\overline{\xi}-\overline{\xi}_0\right)^3\right\}\right] \\ &\oplus \cdots\end{aligned}$$

That is,

$$\begin{aligned}\left[\underline{\eta}(0.1), \overline{\eta}(0.3)\right] &= [\alpha-1, 1-\alpha] \\ &\oplus \frac{\Gamma(2-\beta)}{\Gamma(2)} \left(\left[-2\alpha^2 + 9\alpha - 7, 2\alpha^2 - 9\alpha + 7\right] \otimes [0.9\alpha - 0.7, 1.1 - 0.9\alpha]\right) \\ &\oplus \cdots\end{aligned}$$

The approximate value of ${}^{\alpha}\widetilde{0.2} = \left[\underline{\eta}(0.1), \overline{\eta}(0.3)\right]$ are describe in below tables. Solution of $\tilde{\eta}\left(\widetilde{0.2}\right)$ are calculated using the MATLAB. The Table 2 describe the lower α−cut of η (i.e., $\underline{\eta}$) for different values of α and β and Table 3 represent the upper α−cut of η (i.e., $\overline{\eta}$) for different values of α and β, respectively. The graphical representation of the solutions are shown in later.

Remarks 2. Table 2 represented the value of lower α−cut of the solution of $\eta\left(\tilde{\xi}\right)$ at the point $\tilde{\xi} = \widetilde{0.2}$ and the value of β trends to 1. Similarly, Table 3 represented the value of upper α−cut of the solution of $\eta\left(\tilde{\xi}\right)$ at the point $\tilde{\xi} = \widetilde{0.2}$ and the value of β trends to 1.

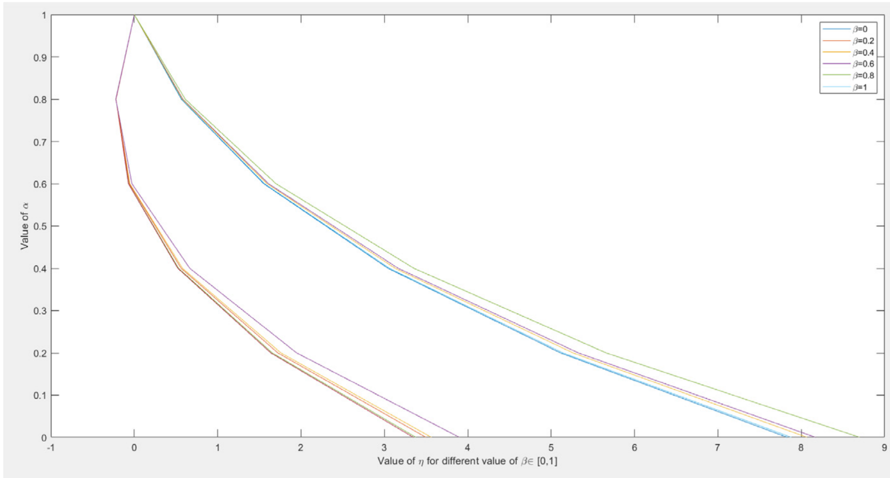

Fig. 2. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 2D.

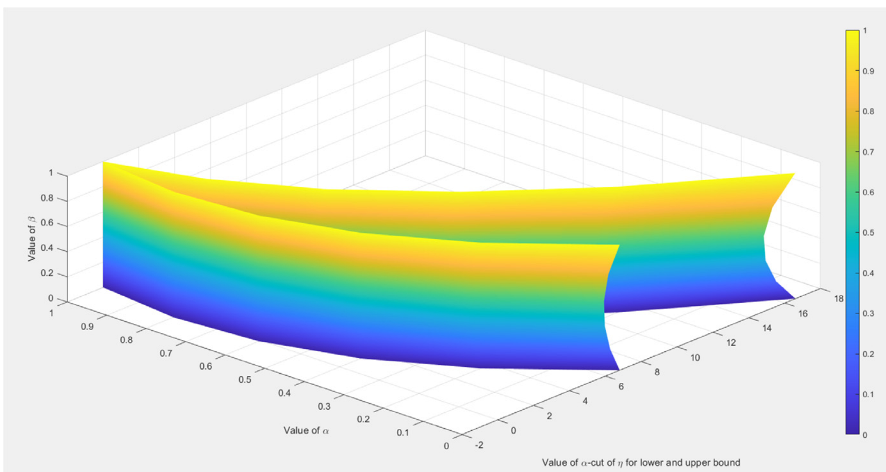

Fig. 3. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 3D from 45° angle.

Remarks 3. The solution of Example 2 are calculated and the geometric representation are shown in Figs. 2–4. The Fig. 2 are pictured the 2D diagram of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ value for the different values of $\alpha \,\&\, \beta$ (i.e., for $\alpha \in [0,1]$ & $\beta = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ & 1). Similarly, Figs. 3 and 4 are representing the 3D diagram of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ value from different angles for different values of $\alpha \,\&\, \beta$ with $\alpha, \beta \in [0,1]$ on 45° and 135°, respectively. The curve at the Figs. 2–4 represents the solution of fuzzy fractional differential equation in Eq. (26) for distinct $\alpha \,\&\, \beta$ at $\tilde{\xi} = \widetilde{0.2}$.

Example 3. Consider another example of fuzzy fractional differential equation as follows:

$$\frac{d^{\beta}\tilde{\eta}}{d\tilde{\xi}^{\beta}} = \tilde{5} \otimes \tilde{\eta} \oplus \tilde{6} \otimes \tilde{\xi} \tag{29}$$

with initial conditions are $\tilde{\eta}\left(\tilde{0}\right) = \tilde{0}$ and $\tilde{\xi} = \tilde{0}$. Find out the value of $\tilde{\eta}\left(\widetilde{0.3}\right)$.

Solution: To solving Example 3, take the $\alpha-$cut on both side of the Eq. (29), get

$$\frac{{}^{\alpha}d^{\beta}\tilde{\eta}}{d\tilde{\xi}^{\beta}} = {}^{\alpha}\tilde{5} \otimes {}^{\alpha}\tilde{\eta} \oplus {}^{\alpha}\tilde{6} \otimes {}^{\alpha}\tilde{\xi} \tag{30}$$

The $\alpha-$cut of the given data, i.e., the parametric forms are follows ${}^{\alpha}\tilde{5} = [4+\alpha, 6-\alpha]$, ${}^{\alpha}\tilde{6} = [5+\alpha, 7-\alpha]$ and ${}^{\alpha}\tilde{0} = [-1+\alpha, 1-\alpha]$. The $\alpha-$cut of $\widetilde{0.3}$ is ${}^{\alpha}\widetilde{0.3} = [0.2+0.1\alpha, 0.4-0.1\alpha]$. Therefore, the Eq. (30) becomes

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = [4+\alpha, 6-\alpha] \otimes \left[\underline{\eta}, \overline{\eta}\right] \oplus [5+\alpha, 7-\alpha] \otimes \left[\underline{\xi}, \overline{\xi}\right] \tag{31}$$

Substitute the value of $\underline{\xi}_0 = (\alpha-1)$, $\overline{\xi}_0 = (1-\alpha)$, $\underline{\eta}_0 = (\alpha-1)$ and $\overline{\eta}_0 = (1-\alpha)$ in Eq. (31), get

$$\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] = [4+\alpha, 6-\alpha] \otimes [\alpha-1, 1-\alpha] \oplus [5+\alpha, 7-\alpha] \otimes [\alpha-1, 1-\alpha]$$

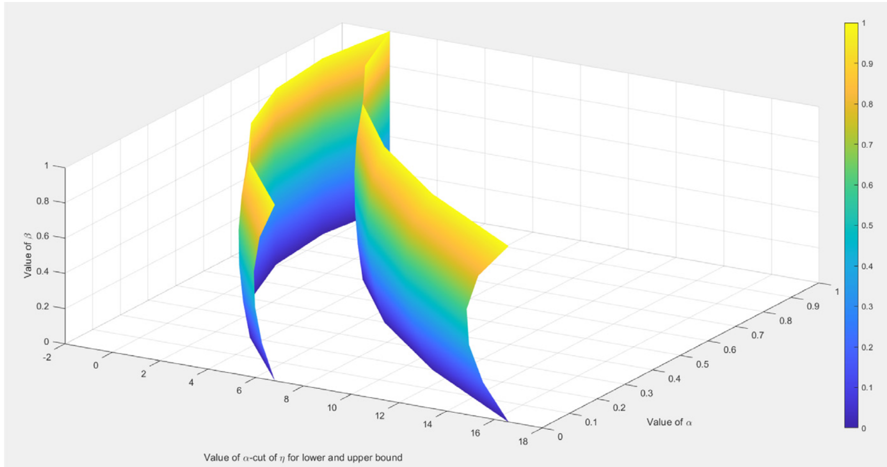

Fig. 4. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 3D from 135° angle.

Table 4
The value of lower $\alpha-$cut of $\underline{\eta}$ for different values of α and β.

α/β	0	0.2	0.4	0.6	0.8	1
0	6.80000000000000	3.43360000000000	1.15680000000000	−0.11680000000000	−0.47360000000000	−0.0000000000000009
0.2	6.26479341364589	3.14310633282196	1.03625500885809	−0.13623211605840	−0.45482659974019	−0.0000000000000008
0.4	5.96941972444398	2.98278658274437	0.96972776562862	−0.14695645308173	−0.44446579956511	−0.0000000000000008
0.6	5.92065777652399	2.95632009778102	0.95874507458940	−0.14872688688313	−0.44275538046884	−0.0000000000000008
0.8	6.16171619071813	3.08715918782363	1.01303884664790	−0.13997461215239	−0.45121096792057	−0.0000000000000008
1	6.80000000000000	3.43360000000000	1.15680000000000	−0.11680000000000	−0.47360000000000	−0.0000000000000009

Table 5
The value of lower $\alpha-$cut of $\overline{\eta}$ for different values of α and β.

α/β	0	0.2	0.4	0.6	0.8	1
0	16.60000000000000	11.08160000000000	6.74880000000000	3.51520000000000	1.29440000000000	0.00000000000000
0.2	15.52958682729180	10.37611537971050	6.32689253100332	3.30144672335765	1.21930639896078	0.00000000000000
0.4	14.93883944888800	9.98676741523632	6.09404717970015	3.18347901610101	1.17786319826045	0.00000000000000
0.6	14.84131555304800	9.92249166603962	6.05560776106291	3.16400424428558	1.17102152187537	0.00000000000000
0.8	15.32343238143630	10.24024374185740	6.24563596326765	3.26027926632373	1.20484387168230	0.00000000000000
1	16.60000000000000	11.08160000000000	6.74880000000000	3.51520000000000	1.29440000000000	0.00000000000000

That is,

$$\begin{aligned}\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right] &= \left[-\alpha^2+7\alpha-6, \alpha^2-7\alpha+6\right] \oplus \left[-\alpha^2+8\alpha-7, \alpha^2-8\alpha+7\right] \\ &= \left[-2\alpha^2+15\alpha-13, 2\alpha^2-15\alpha+13\right]\end{aligned}$$

Now put the value of fractional derivative $\left[\frac{d\underline{\eta}^{\beta}}{d\underline{\xi}^{\beta}}, \frac{d\overline{\eta}^{\beta}}{d\overline{\xi}^{\beta}}\right]$ in fuzzy factional Taylor's theorem, get

$$\begin{aligned}\left[\underline{\eta}(0.2), \overline{\eta}(0.4)\right] &= [\alpha-1, 1-\alpha] \\ &\oplus \frac{\Gamma(2-\beta)}{\Gamma(2)}\left(\left[-2\alpha^2+15\alpha-13, 2\alpha^2-15\alpha+13\right] \otimes [0.9\alpha-0.6, 1.2-0.9\alpha]\right) \\ &\oplus \cdots \\ &= [\alpha-1, 1-\alpha] \\ &\oplus \frac{\Gamma(2-\beta)}{\Gamma(2)}\left[-1.8\alpha^3+14.7\alpha^2-20.7\alpha+7.8, -1.8\alpha^3+15.9\alpha^2-29.7\alpha+15.6\right] \\ &\oplus \cdots\end{aligned}$$

Approximate value of the ${}^{\alpha}\widetilde{0.3} = \left[\underline{\eta}(0.2), \overline{\eta}(0.4)\right]$ are describe in below tables calculated by MATLAB to calculated $\tilde{\eta}\left(\widetilde{0.2}\right)$ from the fuzzy fractional differential equation represent in Eq. (29). The Tables 4 and 5 represented the lower $\alpha-$cut of η (i.e., $\underline{\eta}$) and the upper $\alpha-$cut of η (i.e., $\overline{\eta}$) for different values of α and β, respectively. The graphical representation of the solutions are shown in later.

Remarks 4. The value of lower $\alpha-$cut of the solution of $\eta\left(\tilde{\xi}\right)$ at the point $\tilde{\xi} = \widetilde{0.3}$ are represented in Table 4 for the value of β trends to 1. Similar way, the value of upper $\alpha-$cut of the solution of $\eta\left(\tilde{\xi}\right)$ at the point $\tilde{\xi} = \widetilde{0.3}$ are shown in Table 5 for the value of β trends to 1.

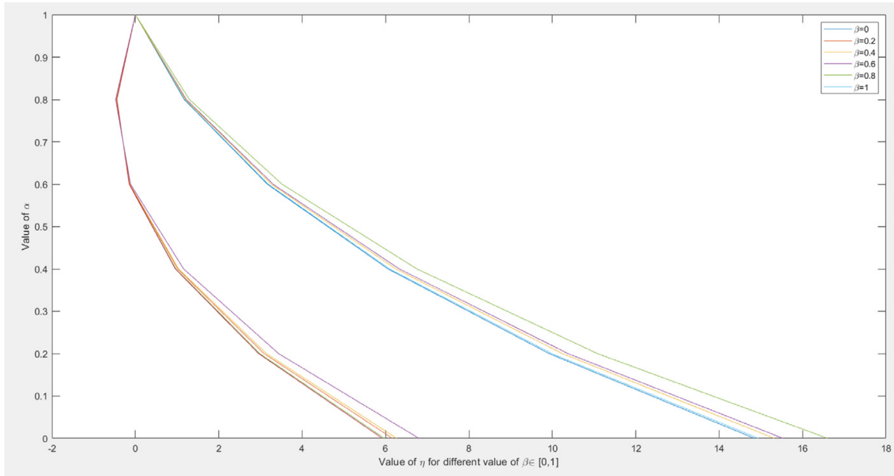

Fig. 5. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 2D.

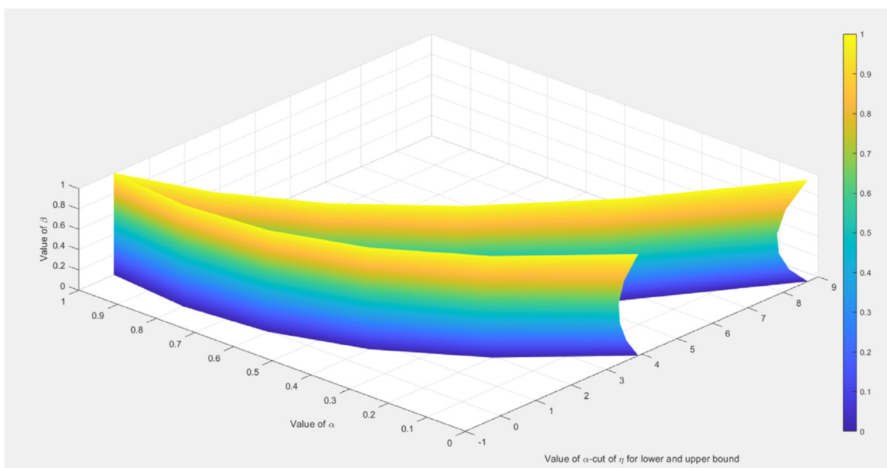

Fig. 6. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 3D from 45° angle.

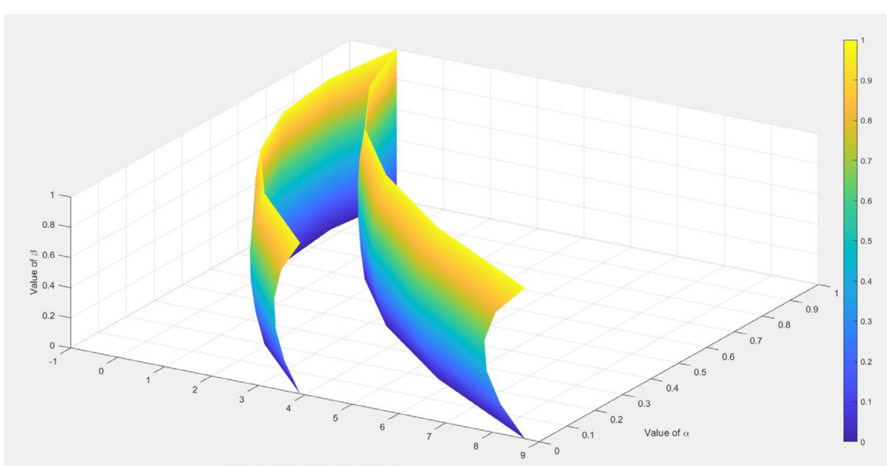

Fig. 7. The solution of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ in parametric form in 3D from 135° angle.

Remarks 5. The Figs. 5–7 are shown the graphical structure of the solution of Example 3. The 2D diagram of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ value for the different values of $\alpha\,\&\,\beta$ (i.e., for $\alpha \in [0,1]$ & $\beta = 0$, 0.2, 0.4, 0.6, 0.8 & 1) are pictured at Fig. 5. The 3D diagram of ${}^{\alpha}\tilde{\eta}\left(\tilde{\xi}\right)$ value from different angles for different values of $\alpha\,\&\,\beta$ with $\alpha, \beta \in [0,1]$ are represented in Figs. 6 and 7 on 45° and 135°, respectively. The solution of fuzzy fractional differential equation in Eq. (29) are fitted in curve shown in Figs. 5–7 for distinct $\alpha\,\&\,\beta$ at $\tilde{\xi} = \widetilde{0.3}$.

5. Result & discussion

The Examples 2 & 3 are represent the fuzzy fractional differential equations and its solutions in fuzzy environment. The Figs. 2–4 & Figs. 5–7 are represents the graphical structure of solutions of Examples 1 & 2, respectively. From above Examples conclude that the dependent variable $\tilde{\eta}$ satisfies fuzzy solution in parametric form with the independent variables α and β which are more visualized in 3D diagrams. The rate of changes of $\tilde{\eta}$ are represented in especially structural way on 3D pictures with the change of $\alpha-$ cuts and β values, simultaneously.

This method has restriction as time increase, value of dependent goes to negative side. Therefore, proposed technique is useful for small time interval. The solution of fuzzy fractional differential equations is evaluated in fuzzy fractional power series with the help of Taylor's expansion. In this study, consider the Caputo fractional derivative of the differential equations. All the properties, examples and solutions with graphs are shown in previous sections.

6. Conclusion

The numerical solution of the differential equation enters in a picture, when explicit analytical solution cannot be obtained. In the other hand, fuzzy fractional differential references the influences of both the memory and uncertainty in a dynamical system. So, when analytical solution for the fuzzy fractional differential equation cannot be obtained, numerical solution may be traced using the approach introduced in this paper. This paper has introduced the fuzzy fractional power series with Taylor's expansion and its convergence criteria. Also, the fuzzy fractional differential equation is solved using that fuzzy fractional power series expansion. Thus, this paper has discussed a possible handful numerical method to deal with the complicated phenomenon where fractional differential equation is studied under fully fuzzy environment.

This study conducted on the fuzzy fractional differential equation and there is some of shortcomings in this research. This study conducted only on fractional order. The whole analysis is based on Taylor's expansion and does not consider other types of power series like the Laurent series, Fourier series, Laplace series, etc. On other hand, this study considered only Caputo type derivative.

6.1. Future research scope

In this study solve the fuzzy fractional power series using Taylor's expansion and examine its convergence criteria. This study may extend further in several branches in future. The proposed theory can be extended in future in the following directions:

(a) In this paper a single fuzzy fractional differential equation is taken into concern. In future, the system of fuzzy fractional differential equations with strong motivations from real world modelling purposes can be considered in the framework of the proposed theory.
(b) The radius of convergence, circle of convergence and other interesting topics in integer order calculus can be revisited in future. This may create a new genre to fuzzy fractional calculus in complex number systems also.
(c) Different fuzzy sets may be considered in future, like type-2 fuzzy set, intuitionistic fuzzy set or neutrosophic fuzzy set for dealing with ambiguity. Also, the solution can apply multiple numerical examples.
(d) Several economic activities consist memory and vagueness and is tried to discuss in terms of fuzzy fractional differential equation. In those cases, where obtaining explicit solution is a formidable task, the proposed technique in this paper can be used to deal economic problems.

Acronym:

2D: Two Dimension
3D: Three Dimension
ABC-TFFD: Atangana–Baleanu–Caputo Type Fuzzy Fractional Derivative
CTFFD: Caputo Type Fuzzy Fractional Derivative
CTFFDGD: Caputo-Type Fuzzy Fractional Derivative with Generalized Differentiability
DTFFD: Differentiable Type Fuzzy Fractional Derivative
FDE: Fuzzy Differential Equation
FPS: Fuzzy Power Series
JTFFD: Jumarie Type Fuzzy Fractional Derivative
$\mathbb{R}$: Set of Real Numbers
PDE: Partial Differential Equation
TFS: Triangular Fuzzy Set
YTFFD: Yang-Type Fuzzy Fractional Derivative

Funding

There is no research funding in this study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors are very grateful to the Editor-in-Chief and anonymous reviewers for their valuable comments and useful suggestions, which improved the quality of this paper.

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