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LONE WOLF THEOREM FOR ONE-SIDED MATCHING PROBLEMS WITH OUTSIDE OPTION *

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Abstract

In this paper, we study one-sided matching problems (so-called roommate problems) with the outside option. In the classical roommate problems, remaining single is conceived as the outside option. However, there are many real life applications where this is not the case. We study roommate problems in which the outside option is defined as having no room. In this general framework, we discuss the generalization of so-called "Lone Wolf Theorem" which states that any agent who is single in one stable matching is single in all other stable matchings. In this study, we show that for the general model with outside option Lone Wolf Theorem still holds.

Keywords: Roommate Problems, Individual Rationality, Outside Option, Capacity Constraint, Lone Wolf Theorem

Dış Mekan Seçeneğinin Olduğu Tek Taraflı Eşleşme Problemlerinde Lone Wolf Teoremi

Öz

Bu makalede dış mekan seceneğinin olduğu tek taraflı eşleşme problemlerini (oda arkadaşı problemlerini) çalışıyoruz. Klasik oda arkadaşı problemlerinde yalnız kalmak dış mekan seçeneği olarak tasarlanmıştır. Ancak, durumun böyle olmadığı birçok uygulama vardır. Biz dış mekan seçeneğinin hiçbir odaya sahip olmamak olarak tanımlandığı oda arkadaşı problemlerini çalışıyoruz. Bu genel çerçevede "Lone Wolf Teoremi" olarak adlandırılan teoremin genelleştirilmesini ele alıyoruz. Bu teorem, durağan bir eşleşmede yalnız kalan bir kişinin diğer tüm durağan eşleşmelerde de yalnız kaldığını belirtir. Bu çalışmada, dış mekan seçeneğinin olduğu genel modelde Lone Wolf Teorem' in hala geçerli olduğunu gösteriyoruz.

Anahtar Sözcükler: Ev Arkadaşı Problemleri, Bireysel Rasyonellik, Dış Mekan Seçeneği, Kapasite Kısıtı, Lone Wolf Teoremi

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Lone Wolf Theorem for One-Sided Matching Problems with Outside Option[1]

Introduction

One-sided matching problems that are known as roommate problems were first introduced by Gale and Shapley (1962). For a matching stability is a fundamental property. Gale and Shapley (1962) showed that for some roommate problems there are no stable matchings. There are mainly three attemps to study roommate problems. The first one is finding necesary and sufficient conditions on the preferences for the existence of stable matchings, Tan (1991) and Chung (2000). The second one is directly analyzing the core for solvable domains where its existence is guarenteed ((Can and Klaus (2013), Klaus (2011), Klaus (2017)). Thirdly, the other attempt is finding other solution concepts rather than the core; such as maximum stable matchings (Tan (1990)), almost stable matchings (Abraham et al. (2006)), P-Stable matchings (Inarra et al. (2008)), stochastically stable matchings (Klaus et al. (2010)), absorbing matchings (Inarra et al. (2013)).

The first condition for stability is individual rationality. In the classical roommate problems, a matching is individually rational if no agent prefers him/herself to its current mate. So, in the classical roommate problems, being single is seen as outside option. But for several cases, being single may not be an outside option. For instance, when allocating offices to professors the outside option is finding a job in a other university. Alternatively, when grouping students of size at most two for a project, being single is not the outside option it means that doing the project by him/herself. In a recent paper of Nizamogullari and Özkal-Sanver (2017), they study roommate problems with capacity where the outside option is introduced into the classical roommate problems and the number of rooms are limited. In roommate problems with capacity agents can either be matched as pairs or remain single or be matched with the outside option with limited number of rooms. In this

1 I thank İpek Özkal-Sanver for her valuable comments.

paper, we study roommate problems with outside option but capacity constraint is not binding.

In many works for roommate problems as in many economic models, Bracing Lemma plays an important role as a consistency result. For instance, Toda (2006) proved Bracing Lemma for many-to-one matching problems. Klaus and Klijn (2010) showed Bracing Lemma for solvable roommate problems.[2] To show this result researchers utilize the Lone Wolf Theorem which is the generalization of "Rural Hospital Theorem". The Lone Wolf Theorem states that any agent who is single in one stable matching is single in all other stable matchings. Gustfield and Irving show the Lone Wolf Theorem by using algorithm that is introduced in Irving (1985) for roommate problems. Klaus and Klijn (2010) give an another proof for this theorem by using bi-choice graphs. İnal (2014) generalizes this result by giving a non-constructive proof.

The aim of the paper is to analyse "The Lone Wolf Theorem" for roommate problems with outside option.

1. The Formal Model and the Basic Axioms

The formal model mostly follows Nizamogullari and Özkal-Sanver (2017). We write **A** for the universal set of agents. A society A is a nonempty and finite subset of **A**. Let n be the number of rooms, i.e., the capacity. We assume that all rooms are identical; furthermore we assume that $|A|/2 \leq n \leq |A|$.[3]

Let c_0 denote the outside option. Each agent $i \in A$ has strict preferences over A and c_0; we denote $\tilde{P}$ for this preference profile. A (roommate) problem (with capacity) is a triple $\tilde{p}=(A, \tilde{P}, n)$. Let $\mathbf{P}_C$ denote the set of all problems with capacity.

A matching for a set of agents A is a function $\mu: A \rightarrow A \cup \{c_0\}$ such that for all $j,k \in A$, $\mu(j)=k$ implies $\mu(k)=j$. Note that $\mu(j)=j$ means that agent j stays in a room as a single; $\mu(j)=c_0$ means that agent j is assigned to no room. For any agent $i \in A$, if $\mu(i)=j$ for some $j \in A$, then (i,j) is called a matched pair under μ.[4]

Let $\mathbf{M}(A,c_0)$ denote the set of all matchings for A and $\mathbf{M}(A) \subseteq \mathbf{M}(A,c_0)$ denote the set of matchings for A in which $\mu^{-1}(c_0)=\emptyset$.

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2 For a detailed analysis of Bracing Lemma for other economic models see Thomson (2015).

3 where |A| denotes the number of the agents in A.

4 Note that j can be i, in which case i is matched with himself/herself under μ.

In the model with capacity, a matching is individually rational if no agent prefers the outside option to its current mate. Formally, a matching $\mu\in$ **M** (A,c_0) is individually rational for a problem $\tilde{p}=(A,\tilde{P},n)$ if and only if for all $i\in A$, $\mu(i)\ \tilde{P}_i\ c_0$. Let $\mathbf{I}(\tilde{p})\subseteq\mathbf{M}(A,c_0)$ denote the set of all such matchings.

To simplfy the notation, we denote $o_{A,\mu}$ for the number of the rooms occupied by one or two agents all belonging to the set of agents in A. More formally, given any set of agents A and any matching $\mu\in\mathbf{M}(A,c_0)$, the number of occupied rooms equals $o_{A,\mu}=|\{i\in A|\mu(i)=i\}|+((|\{i\in A|\mu(i)=j \text{ and } j\in A/\{i\}\}|)/2)$. Note that given any subset $A'\subset A$, we have $o_{A,\mu}=o_{A',\mu}+o_{A\backslash A',\mu}$, only if $\cup_{i\in A}\{\mu(i)\}=A'\cup\{c_0\}$.

An agent blocks a matching if the agent prefers staying single to its current mate, and there is an available room. More formally, an agent i blocks a matching $\mu\in\mathbf{M}(A,c_0)$ for a problem $\tilde{p}=(A,\tilde{P},n)$ if and only if $i\ \tilde{P}_i\ \mu(i)$ and $o_{A,\mu}<n$. Let $\mathbf{B}(\tilde{p})\subseteq\mathbf{M}(A,c_0)$ denote the set of all such matchings. A matching $\mu\in\mathbf{M}(A,c_0)$ is strong individually rational for $\tilde{p}=(A,\tilde{P},n)$ if and only if it is individually rational for $\tilde{p}$ and there is no agent i blocking $\mu\in\mathbf{M}(A,c_0)$ for $\tilde{p}$. Let $\mathbf{IB}(\tilde{p})=(\mathbf{I}(\tilde{p})\cap\mathbf{B}(\tilde{p}))\subseteq\mathbf{M}(A,c_0)$ denote the set of all such matchings.

In a roommate problem with capacity, a pair of agents blocks a matching when both of them wish so and at least one of the following holds:

(i) One of the blocking agents is single in the room, so that the other agent freely moves to this room.

(ii) Both agents share their rooms, and either there is an available room occupied by nobody or their current mates approve the exchange of mates as they also benefit from.

Formally, a pair of agents (i,j) with $i\neq j$ blocks a matching $\mu\in\mathbf{M}(A,c_0)$ for a problem $\tilde{p}=(A,\tilde{P},n)$ if and only if $j\ \tilde{P}_i\ \mu(i)$ and $i\ \tilde{P}_j\ \mu(j)$ and

- either $\mu(i)=i$ or $\mu(j)=j$ or both
- $\mu(i)\neq i$ and $\mu(j)\neq j$ and $o_{A,\mu}<n$
- $\mu(i)\neq i$ and $\mu(j)\neq j$ and $o_{A,\mu}=n$; $\mu(j)\ \tilde{P}_{\mu(i)}\ i$ and $\mu(i)\ \tilde{P}_{\mu(j)}\ j$.

A matching $\mu\in\mathbf{M}(A,c_0)$ is stable for $\tilde{p}=(A,\tilde{P},n)$ if and only if it is strong individually rational for $\tilde{p}$ and there is no pair (i,j) blocking $\boldsymbol{\mu}\in\mathbf{M}(A,c_0)$ for $\tilde{p}$. Let $\mathbf{S}(\tilde{p})\subseteq\mathbf{M}(A,c_0)$ denote the set of all such matchings.

2. A Model with Outside Option

Throughout this section, letting n=|A|, we assume that the capacity constraint is not binding. Let $\mathbf{P_{SWO}}$ denote the set of all problems in this domain.[5]

Our aim is to show "The Lone Wolf Theorem" for this domain. For the proof of the below theorem the non-constructive proof of İnal (2014) is adapted to the more general roommate problems with outside option.

Theorem 2.1 (The Lone Wolf Thorem) For any problem $\tilde{p}$=(A, $\tilde{P}$,n) ∈ $\mathbf{P_{SWO}}$, for all μ, μ′∈$\mathbf{S}(\tilde{p})$ we have μ(i)∈{i,c_0} if and only if μ′(i)∈{i,c_0}.

Proof. Let $\tilde{p}$=(A, $\tilde{P}$,n) ∈ $\mathbf{P_{SWO}}$ be a problem and μ,μ′∈$\mathbf{S}(\tilde{p})$ be two stable matchings for $\tilde{p}$. Suppose for a contradiction there is an agent i_1∈A with μ′(i_1)∈{i_1,c_0} but μ(i_1)∉{i_1,c_0}. Then there is an agent i_2∈A\{i_1,c_0} such that μ(i_1)=i_2. Now, we will prove the following claim.

Claim: For each k∈$\mathbb{Z}^+$ there exists i_{k+1}∈A\{c_0,i_1,i_2,...,i_k} such that

- i_{k+1}=μ(i_k) and μ(i_k) $\tilde{P}_{i_k}$ μ′(i_k) if k is odd
- i_{k+1}= μ′ (i_k) and μ′(i_k) $\tilde{P}_{i_k}$ μ(i_k) if k is even.

Proof of the Claim: The proof is by induction on k.

Let k=1. Since μ∈$\mathbf{S}(\tilde{p})$ we have μ∈$\mathbf{IB}(\tilde{p})$=($\mathbf{I}(\tilde{p})$∩$\mathbf{B}(\tilde{p})$) which implies μ(i_1) $\tilde{P}_{i_1}$ μ′(i_1).

Let k=2. Since μ′∈$\mathbf{S}(\tilde{p})$ we have μ′(i_2) $\tilde{P}_{i_2}$ μ(i_2) otherwise the pair (i_1,i_2) blocks the matching μ′ for $\tilde{p}$. Since μ∈$\mathbf{S}(\tilde{p})$ we have μ′(i_2)∉{c_0,i_2}. Also, since $\tilde{P}_i$ is strict μ′(i_2)≠i_1. So there is an agent i_3∈A\{c_0,i_1,i_2} with i_3=μ′(i_2) and μ′(i_2) $\tilde{P}_{i_2}$ μ(i_2)

Assume that for some k′∈$\mathbb{Z}^+$ and k∈$\mathbb{Z}^+$ with k≤k′, there exists i_{k+1}∈A\{c_0,i_1,i_2,...,i_k} such that

- i_{k+1}=μ(i_k) and μ(i_k) $\tilde{P}_{i_k}$ μ′(i_k) if k is odd
- i_{k+1}= μ′ (i_k) and μ′(i_k) $\tilde{P}_{i_k}$ μ(i_k) if k is even.

Now, show for k′+1. By induction assumption if k′ is odd there is $i_{k'+1}$∈A\{c_0,i_1,i_2,...,$i_{k'}$} with $i_{k'+1}$=μ($i_{k'}$) and μ($i_{k'}$) $\tilde{P}_{i_k}$ μ′($i_{k'}$). Since μ′∈$\mathbf{S}(\tilde{p})$, we have μ′($i_{k'+1}$) $\tilde{P}_{i_{k'+1}}$ μ($i_{k'+1}$), otherwise the pair ($i_{k'}$,$i_{k'+1}$) blocks the matching μ′. Since μ∈$\mathbf{S}(\tilde{p})$, we have μ′($i_{k'+1}$)∉{c_0,$i_{k'+1}$}. Also since $\tilde{P}_i$ is strict, μ′($i_{k'+1}$)≠$i_{k'}$.

5 Note that, on the domain of $\mathbf{P_{SWO}}$ there is no restriction on the preferences of the agents. So, $\mathbf{P_{SWO}}$ is a larger domain than the domain that is defined in Nizamogullari and Özkal-Sanver (2017) as $\mathbf{P_S}$, that is, $\mathbf{P_S}$ ⊆ $\mathbf{P_{SWO}}$.

Since $\mu,\mu' \in \mathbf{M}(A,c_0)$, we have $\mu'(i_{k'+1}) \notin \{i_1,i_2,...,i_{k'-1}\}$. So there is an agent $i_{k'+2} \in A \setminus \{c_0,i_1,i_2,...,i_{k'+1}\}$ with $i_{k'+2}=\mu'(i_{k'+1})$ and $\mu'(i_{k'+1})\ \tilde{P}_{i_{k'+1}}\ \mu(i_{k'+1})$. Note that, if k' is odd then $k'+1$ is even which proves induction hypothesis for $k'+1$ is even.

Now, if k' is even, then by induction assumption there is $i_{k'+1} \in A \setminus \{c_0,i_1,i_2,...,i_{k'}\}$ with $i_{k'+1}=\mu'(i_{k'})$ and $\mu'(i_{k'})\ \tilde{P}_{i_{k'}}\ \mu(i_{k'})$. Since $\mu \in \mathbf{S}(\tilde{p})$), we have $\mu(i_{k'+1})\ \tilde{P}_{i_{k'+1}}\ \mu'(i_{k'+1})$, otherwise the pair $(i_{k'},i_{k'+1})$ blocks the matching μ. Since $\mu' \in \mathbf{S}(\tilde{p})$), we have $\mu(i_{k'+1}) \notin \{c_0,i_{k'+1}\}$. Also since $\tilde{P}_i$ is strict, $\mu(i_{k'+1}) \neq i_{k'}$. Since $\mu,\mu' \in \mathbf{M}(A,c_0)$, we have $\mu(i_{k'+1}) \notin \{i_1,i_2,...,i_{k'-1}\}$. So, there is an agent $i_{k'+2} \in A \setminus \{c_0,i_1,i_2,...,i_{k'+1}\}$ with $i_{k'+2}=\mu(i_{k'+1})$ and $\mu(i_{k'+1})\ \tilde{P}_{i_{k'+1}}\ \mu'(i_{k'+1})$. Note that, if k' is even then $k'+1$ is odd which proves the induction hypothesis for $k'+1$ is odd.

Therefore, for each $k \in \mathbb{Z}^+$ there exists $i_{k+1} \in A \setminus \{c_0,i_1,i_2,...,i_k\}$, but A is finite which gives a contradiction.

Remark: The Lone Wolf Theorem is not true when capacity constraint is binding. To see that consider the following example:

Example: Consider the problem $\tilde{p}=(A,\tilde{P},n) \in \mathbf{P_C}$ where $A=\{a,b,c,d\}$, $n=2$, and $\tilde{P}$ as follows:

$\tilde{P}_a$	$\tilde{P}_b$	$\tilde{P}_c$	$\tilde{P}_d$
b	a	c	c
…	…	d	a
…	…	c_o	b
…	…	…	c_o

Consider the matchings $\mu=\{(a,b),(c,d)\}$ and $\mu'=\{(a,b),(c),(d,c_0)\}$. It is obvious that $\mu,\mu' \in \mathbf{I}(\tilde{p})$. Now, we show that $\mu,\mu' \in \mathbf{B}(\tilde{p})$. Since $o_{A,\mu}=n$, the agent c can not block the matching μ for $\tilde{p}$ implying that $\mu \in \mathbf{B}(\tilde{p})$. Similary, since $o_{A,\mu'}=n$, the agent d can not block the matching μ' for $\tilde{p}$ implying that $\mu' \in \mathbf{B}(\tilde{p})$. Also, there is no pair of agents that blocks μ for $\tilde{p}$. Since for the agent c we have $c\,\tilde{P}_c\,d$, the pair c and d can not block the matching μ' for $\tilde{p}$. Hence, $\mu,\mu' \in \mathbf{S}(\tilde{p})$. But μ and μ' do not have the same set of agents that are single or are matched with the outside option.

This example establishes that we can not show Bracing Lemma by using Lone Wolf Theorem on the domain of roommate problems with capacity.

Concluding Remarks

The aim of the paper was to generalize the Lone Wolf Theorem to roommate problems with outside option. We consider one-sided matching problems with the outside option In this model, the Lone Wolf Theorem holds, that is any agent who is single or is matched with the outside option in one stable matching is again single or is matched with the outside option in all other stable matchings. Next, we consider the general domain with capacity constraint. For this general domain, due to the capacity constraint the Lone Wolf Theorem does not hold. As one can see that the stable matching μ' in Example 2.1 is not fair we have to define some fairness axioms for the stable matchings. Our further study on the agenda is after defining these axioms, to study new solution concept and to analyse the Lone Wolf Theorem for this solution concept.

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