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The largest orbit sizes of linear group actions and abelian quotients

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ABSTRACT

Let G be a finite group and let V be a finite and faithful G -module such that V is completely reducible, possibly of mixed characteristic, or V is a G -module over a field of characteristic p with $O_p(G) = 1$. Suppose that M is the largest orbit size in the action of G on V . It is known that the index of the commutator subgroup G' in G is bounded by M , that is, $|G : G'| \leq M$. In this paper, we classify all linear group actions in which $|G : G'| = M$. It turns out that our classification is a vast generalization of a classic 1967 result by D. Passman. We also include an application of the main results to Brauer's $k(B)$ -problem.

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1. Introduction

One important aspect of finite linear groups is the study of their orbits. Maximal orbit sizes play an important role in understanding the structure of a finite group action and

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are related to various important concepts such as conjugacy classes, representations, and character tables.

On the other hand, the interplay between the abelian quotient G/G' of a finite group and various invariants of finite group actions have been an object of study for a long time. One of the notable contributions in this area comes from the work of Kovacs and Praeger [13]. They presented an upper bound on the orders of abelian quotients of permutation groups as a function of their degrees, and characterized the specific groups that achieve these bounds. Moreover, Aschbacher and Guralnick studied abelian quotients of permutation groups as well and, in addition, proved under mild conditions that the index of the commutator subgroup of a linear group G is strictly less than $|V|$, where V is the vector space on which G acts naturally [2].

The relationship between abelian quotient groups and maximal orbit sizes was first explored by the second author and Yang in [12]. They showed that if M is the largest orbit size in the action of a finite solvable group G on finite faithful completely reducible G -module V , then the index of the commutator subgroup G' in G is bounded by M , that is $|G : G'| \leq M$ ([12]); they later extended this bound to arbitrary groups ([11]). For solvable groups, they obtained more specific information, which in [18] was extended to arbitrary groups. This result has some implications for understanding the structure and behavior of finite groups on the faithful completely reducible vector spaces. For example, the extreme case $|G : G'| = M$ of this result has been studied extensively in a collection of papers [4,9,10]. In [9] and [10], all nonabelian group actions in which $|G : G'| = M$ were classified under the further assumption that V is an irreducible G -module and G has exactly two or three different orbits of size M on V , respectively. Afterwards, the authors of this paper relaxed the assumption of irreducibility on V in the previous results to complete reducibility, and they classified all linear group actions in which $|G : G'| = M$ and G is a nonabelian group having at most three orbits of the largest size M on V . In this context, one interesting problem is to completely classify all linear group actions if we are in the case $|G : G'| = M$ and G is nonabelian. The following results answer this question by classifying all such group actions. These theorems provide a complete understanding of the structure and properties of the groups that have such linear actions on $V = V(n, q)$, where $V(n, q)$ denotes the n -dimensional vector space over the field with q elements.

Theorem A. *Let G be a finite nonabelian group and let V be a finite faithful irreducible G -module. Suppose that $|G : G'|$ is the largest orbit size in the action of G on V . Then one of the following situations occurs.*

- (a) G is D_8 and $V = V(2, 3)$;
- (b) G is the central product of D_8 and C_4 , and $V = V(2, 5)$ or
- (c) G is the central product of D_8 and Q_8 , and $V = V(4, 3)$.

Note that, as special cases, Theorem A reconfirms Conjecture 4.1 in [9] and confirms Conjecture 1 in [10]. The reader might also note that the group actions in the conclusion

of the theorem are exactly the ones which occur in the conclusion of a result of Passman, see [16, Lemma 1.5(i)], where he characterized linear actions without regular orbits of groups which are either extraspecial of exponent p , or a central product of an extraspecial 2-group and the cyclic group of order 4. From this perspective Theorem A can be viewed as a generalization of Passman's result. In fact, it was this similarity of the group actions found/conjectured in [4,9–11] with those in Passman's result that ultimately guided the authors to conjecturing the statement of Theorem A, and it also will be useful in its proof.

The result being discussed concerns finite faithful irreducible G -modules, and we next present the following classification for finite faithful completely reducible G -modules, possibly of mixed characteristic.

Theorem B. *Let G be a finite group and let V be a finite faithful completely reducible G -module with $C_V(G) = 0$, possibly of mixed characteristic. Suppose that $|G : G'|$ is the largest orbit size in the action of G on V . Then*

$$G = \left(\bigtimes_{i=1}^{n_1} E_i \times \bigtimes_{j=1}^{n_2} F_j \times \bigtimes_{l=1}^{n_3} K_l \right) \times A =: N \times A$$

and

$$V = \left(\bigoplus_{i=1}^{n_1} V_i \oplus \bigoplus_{j=1}^{n_2} U_j \oplus \bigoplus_{l=1}^{n_3} W_l \right) \oplus W =: U \oplus W,$$

where n_1, n_2, n_3 are non-negative integers, and $E_i \cong D_8$, $F_j \cong D_8 \circ C_4$, $K_l \cong D_8 \circ Q_8$ and A is an abelian group, $V_i \cong V(2, 3)$, $U_j \cong V(2, 5)$ and $W_l \cong V(4, 3)$ for $i \in \{1, \dots, n_1\}$, $j \in \{1, \dots, n_2\}$ and $l \in \{1, \dots, n_3\}$.

Furthermore, N acts faithfully on U and trivially on W , and A acts trivially on U and faithfully on W . Moreover, N acts on U as follows: Each E_i , F_j , K_l acts faithfully on their respective modules V_i , U_j , W_l , and trivially on all the other modules.

Theorem B looks exactly like one naïvely would expect an extension of Theorem A to completely reducible modules to look like. Its proof, however, needs some effort. The same is true for the extension of Theorem A to another common generalization of irreducible modules as follows.

Theorem C. *Let G be a finite group and let V be a finite faithful G -module over a finite field of characteristic p with $C_V(G) = 0$. Suppose that $O_p(G) = 1$. If $|G : G'|$ is the largest orbit size in the action of G on V , then one of the following situations holds.*

- (a) G and V are as in Theorem B with $n_2 = 0$.
- (b) G and V are as in Theorem B with $n_1 = 0$ and $n_3 = 0$.

Remark 1.1. It is easy to see that in Theorem B the number of orbits of size $|G : G'|$ equals

$$2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot s,$$

where s is the number of regular orbits of A on W (note that $s \geq 1$). A similar remark applies to Theorem C.

When working with a finite group G and a field k of prime characteristic p , where p divides $|G|$, modular representation theory over k involves analyzing the summands of the group algebra $k[G]$, which are called blocks and are denoted by B . Each block B is associated with a p -subgroup of G called the defect group denoted by D , which is uniquely determined up to conjugation. It has been conjectured by Brauer that the inequality $k(B) \leq |D|$ holds, where $k(B)$ is the number of irreducible characters of G in B . If B_D is a Brauer correspondent of B in $DC_G(D)$, then the notation $N_G(D, B_D)$ denotes the elements in the group G that stabilize D and B_D under conjugation. The inertial quotient of B is defined by $N_G(D, B_D)/C_G(D)$ and we will use the notation $I(B)$ for the inertial quotient of B . As an application of Theorems A and B, we shall present the following result, which extends a result of B. Sambale (e.g. [20, Lemma 14.7]).

Corollary D. *Let B be a block of a finite group with abelian defect group D . Write $I(B)$ for the inertial quotient of B , and suppose that its commutator subgroup satisfies $|I(B)'| \leq 7$. Then Brauer's $k(B)$ -conjecture $k(B) \leq |D|$ holds.*

We have adopted standard notations in this paper and have provided explanations wherever necessary. It is convenient here to mention the semilinear group. If V is a finite vector space of dimension n over $\text{GF}(q)$ for a prime power q , we will use the notation $\Gamma(q^n) = \Gamma(V)$ to denote the semilinear group of V . More precisely, the semilinear group is defined by

$$\Gamma(V) = \{x \mapsto ax^\sigma \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times, \sigma \in \text{Gal}(\text{GF}(q^n) : \text{GF}(q))\}.$$

Also, we write $A \circ B$ to denote the central product of the groups A and B .

2. Auxiliary results

In this section, we will present an overview of some auxiliary results, which are essential tools for the proof of our main theorems. We will start to discuss some of the key results related to groups and their actions without giving the proofs, but providing references. The following theorem has been extensively discussed in the introduction of this paper, and is a consequence of a collection of papers.

Theorem 2.1. (Keller, Qian, Yang) [11,12,18] *Let G be a finite group and V a finite faithful completely reducible G -module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V . Then*

$$|G : G'| \leq M.$$

More precisely, we have one of the following:

- (a) $|G : G'| < M$;
- (b) $|G : G'| = M$ and G is abelian; or
- (c) $|G : G'| = M$, G is nilpotent, and G has at least two different orbits of size M on V .

Lemma 2.2. [12, Lemma 2.1] *Let G be a finite group and $N \trianglelefteq G$. Then*

$$|G/G'| = |G/G'N| \cdot |N : N \cap G'|;$$

and

$$|G : G'| \text{ divides } |G/N : (G/N)'| \cdot |N : N'|$$

We frequently appeal to the next lemma in the rest of this paper.

Lemma 2.3. [4, Lemma 2.4] *Let G be a finite nonabelian group and V a finite faithful completely reducible G -module, possibly of mixed characteristic. Suppose that $|G : G'|$ is the largest orbit size in the action of G on V . If U is a G -submodule of V , then*

$$|G : G'| = |G/C_G(U) : (G/C_G(U))'| \cdot |C_G(U) : C_G(U)'|.$$

Moreover, $|G/C_G(U) : (G/C_G(U))'|$ is the largest orbit size of G on U , and $|C_G(U) : C_G(U)'|$ is the largest orbit size of $C_G(U)$ on a G -invariant complement of U in V .

Theorem 2.4. [7, Theorem 3.2] *Let G be a finite nilpotent group that acts faithfully and irreducibly on a finite vector space V and assume that G' is cyclic. Then there exists a vector $v \in V$ such that $|C_G(v)| \leq 2$.*

We will need the subsequent three lemmas for the proof of our main results. The first lemma is easily obtained as a result of Step 3.2 in the proof of main theorem of [9]. For the sake of completeness, we prove this lemma here.

Lemma 2.5. *Let G be a finite nonabelian p -group for a prime p and let V be a finite faithful irreducible G -module. If $|G : G'|$ is the largest orbit size in the action of G on V , then V cannot be quasiprimitive.*

Proof. Let V be quasiprimitive, and put $M = |G : G'|$. Then G becomes a quasiprimitive linear group and hence every abelian normal subgroup of G is cyclic. Since G is a p -group, G is cyclic, quaternion, dihedral, or semi-dihedral and $G \not\cong D_8$ by Corollary 1.3 of [15]. It is well known that the derived subgroup of the quaternion, dihedral, and semi-dihedral 2-groups have index 4 in G . We also know that there exists a $U \trianglelefteq G$, where U is cyclic and $|G : U| \leq 2$. Thus, U has a regular orbit on V , which gives the inequality $M \geq |U| \geq |G|/2$. Since G is a nonabelian 2-group, we have $|G| \geq 8$, which leads the inequality that

$$M = |G : G'| = 4 \leq |G|/2 \leq M.$$

It follows that $|G'| = 2$ and $|G| = 8$, and so $G \cong Q_8$. It is well known that the quaternion group Q_8 has a regular orbit on V creating a contradiction with $M = 4$. Thus, we conclude that V cannot be quasiprimitive, and in particular, V is imprimitive. $\square$

It is well known that if G is a finite abelian group and V is a finite faithful and completely reducible G -module, possibly of mixed characteristic, then G has a regular orbit on V . The next lemma provides us the existence of regular orbits of G on V , where G is the central product of D_8 and a cyclic 2-group, that is, $G = D_8 \circ C_{2^n}$ for some $n \in \mathbb{N}$.

Lemma 2.6. *Let G be the central product of the dihedral group D_8 and the cyclic group of order 2^n for $n \in \mathbb{N}$ and let V be a finite faithful irreducible G -module. Suppose that $n \geq 3$. Then G has a regular orbit on V .*

Proof. Since $G = D_8 \circ C_{2^n}$, we know that $|Z(G)| = 2^n \geq 8$ and $|G : Z(G)| = 4$. Assume that G has no regular orbit on V , and let $0 \neq v \in V$. Since the action of the cyclic group $Z(G)$ on V is Frobenius, we have $C_G(v) \cap Z(G) = 1$, and hence $C_G(v) \times Z(G) \leq G$. This gives us $|C_G(v)| = 2$, because $|G : Z(G)| = 4$ and G is a nonabelian group. Since v was arbitrary, this shows that $|C_G(v)| = 2$ for all nonzero vectors v in V . Now let H be a subgroup of G such that $H \cong D_8 \circ C_4$. Note that H is characteristic in G , because $H = \Omega_4(G) = \langle g \in G \mid g^4 = 1 \rangle$. In particular, all involutions of G are contained in H , and it is well-known that H has six non-central involutions. Now by [16, Lemma 1.9] we see that $6 = |V|^{1/2} + 1$ which implies that $|V| = 25$. Thus G is a subgroup of a Sylow 2-subgroup P of $\text{GL}(2, 5)$. Thus $P \cong C_4 \wr C_2$ and $|P| = 32$. This forces $n = 3$ and $G = P$. However, the groups $D_8 \circ C_8$ and $C_4 \wr C_2$ are not isomorphic, as is well-known, and this contradiction refutes the assumption of G having no regular orbit and proves the lemma. $\square$

Lemma 2.7. *Let G a finite nonabelian group and let V be a finite faithful irreducible non-quasiprimitive G -module. Suppose that $M = |G : G'|$ is the largest orbit size in the action of G on V . Then there exists a normal subgroup N of G such that $|G : N| = p$ for some prime p , and the following two statements hold:*

- (a) $V_N = V_1 \oplus \dots \oplus V_p$ for irreducible N -submodules V_i of V_N which are faithfully permuted by G/N . Furthermore, $M \geq M_N = M_1 M_2 \geq |N : N'|$, where M_N is the largest orbit size in the action of N on V_N , M_1 is the largest orbit size of N on an irreducible N -submodule U of V_N , and M_2 is the largest orbit size of $C_N(U)$ on a G -invariant complement of U in V_N .
- (b) If G is a 2-group and $C_N(V_i)/C_N(V_i)'$ is not cyclic for $i = 1, 2$, then

$$|(H/S)'| \geq |[g, (C_N(V_1)C_N(V_2))/S]| \geq |C_N(V_i)/C_N(V_i)'| > 1, \quad (1)$$

where $H = \langle g, C_N(V_1)C_N(V_2) \rangle \leq G$ for $g \in G - N$ and $S = C_N(V_1)' \times C_N(V_2)'$.

Proof. Since G is a nonabelian group, we know by Theorem 2.9 of [18] that G is nilpotent. By Corollary 0.3 of [15] there exists a normal subgroup N of G such that $|G : N| = p$ and $V_N = V_1 \oplus \dots \oplus V_p$ for irreducible N -submodules V_i of V_N . Let M_1 be the largest orbit size in the action of $N/C_N(V_1)$ on V_1 and let M_2 be the largest orbit size of $C_N(V_1)$ on $V_2 \oplus \dots \oplus V_p$. To simplify the notation, throughout this proof W will denote $V_2 \oplus \dots \oplus V_p$. Let $x \in V_N$ be in a largest orbit of N on V_N . Then we may write $x = x_1 + x_2$, where $x_1 \in V_1$ and $x_2 \in W$. Then

$$\begin{aligned} |G : G'| = M &\geq M_N = |x^N| = |N : C_N(x)| = |N : C_N(x_1) \cap C_N(x_2)| \\ &= |N : C_N(x_1)| |C_N(x_1) : C_N(x_1) \cap C_N(x_2)| \\ &= |x_1^N| |C_N(x_1) : C_{C_N(x_1)}(x_2)| = |x_1^N| |x_2^{C_N(x_1)}| \end{aligned} \quad (2)$$

Suppose that $|x_1^N| < M_1$ or $|x_2^{C_N(x_1)}| < M_2$. Then there exist $y_1 \in V_1$ and $y_2 \in W$ such that $|y_1^N| = M_1$ and $|y_2^{C_N(V_1)}| = M_2$. This gives us

$$M_N = |x_1^N| |x_2^{C_N(x_1)}| < M_1 M_2 = |y_1^N| |y_2^{C_N(V_1)}| \leq |y_1^N| |y_2^{C_N(y_1)}| = |N : C_N(y_1 + y_2)|,$$

which is a contradiction. This contradiction shows that $|x_1^N| = M_1$ and $|x_2^{C_N(x_1)}| = M_2$, and hence $M \geq M_N = M_1 M_2$. On the other hand, since $N/C_N(V_1)$ acts faithfully on V_1 and $C_N(V_1)$ acts faithfully on W , we have from Theorem 2.1 and Lemma 2.2 that

$$\begin{aligned} M \geq M_N = M_1 M_2 &\geq |N/C_N(V_1) : (N/C_N(V_1))'| |C_N(V_1) : C_N(V_1)'| \\ &\geq |N/C_N(V_1) : (N/C_N(V_1))'| |C_N(V_1) : C_N(V_1) \cap N'| \\ &= |N : N'|. \end{aligned}$$

Thus, we conclude that $M \geq M_N = M_1 M_2 \geq |N : N'|$, and this completes the proof of the case (a).

By case (a), we have that $|G : N| = 2$ and $V_N = V_1 \oplus V_2$ for irreducible N -submodules V_i of V_N . Let $g \in G - N$ and let us define the subgroups $H = \langle g, C_N(V_1)C_N(V_2) \rangle$ and $S = C_N(V_1)' \times C_N(V_2)'$ of G . Clearly, S is a normal subgroup of H . Note that $C_N(V_1)^g = C_N(V_2)$ and $C_N(V_2)^g = C_N(V_1)$. Since $C_N(V_1)/C_N(V_1)'$ is not cyclic by the hypothesis, there exist elements $x_1, x_2 \in C_N(V_1)$ such that $x_1 C_N(V_1)' \neq x_2 C_N(V_1)'$. If

$x_1S = x_2S$, then $x_1^{-1}x_2 \in C_N(V_1) \cap S = C_N(V_1)'$, and hence we get a contradiction that $x_1C_N(V_1)' = x_2C_N(V_1)'$. Thus $x_1S \neq x_2S$, and hence clearly $[x_1S, g] \neq [x_2S, g]$, because $[x_iS, g] = (x_i^{-1}S)(x_i^gS)$ and $x_i^g \in C_N(V_2)$ for $i = 1, 2$. Therefore we have $|C_N(V_1)/C_N(V_1)'|$ mutually distinct elements in $[g, (C_N(V_1)C_N(V_2))/S]$, which gives us the desired inequality (1). This completes the proof of lemma. $\square$

3. Proof of Theorem A

In this section, we will provide a detailed proof of Theorem A.

Theorem 3.1. *Let G be a finite nonabelian group and let V be a finite faithful completely reducible G -module, possibly of mixed characteristic. Suppose that $|G : G'|$ is the largest orbit size in the action of G on V . Then $G = P \times A$, where P is the Sylow 2-subgroup of G and A is an abelian group of odd order.*

Proof. Put $M = |G : G'|$. Since G is a nonabelian group, we know by Theorem 2.9 of [18] that G is nilpotent. We wish to show that the Hall 2'-subgroup of G is an abelian group. To do this, we proceed by induction on $|GV|$, and we first reduce to the case that V is irreducible.

Step 1. A reduction to the case that V is irreducible

Let $V = V_1 \oplus \dots \oplus V_n$, where $n > 1$ and the V_i are the proper nontrivial irreducible G -submodules of V for all $i \in \{1, \dots, n\}$. Since the nonabelian group G is isomorphic to a subgroup of $\bigtimes_{i=1}^n G/C_G(V_i)$, there exists at least one G -module V_i for some $i \in \{1, 2, \dots, n\}$ such that $G/C_G(V_i)$ is nonabelian. Without loss of generality, we may choose $G/C_G(V_1)$ as a nonabelian group. If $C_G(V_1) = 1$, then G acts faithfully on V_1 , and hence $|G : G'| = M$ is also the largest orbit size in the action of G on V_1 by Theorem 2.1. By induction, we get the desired conclusion that H is abelian. So we may assume that $C_G(V_i) > 1$ for all $i = 1, 2, \dots, n$. It follows by Lemma 2.3 that for any i such that $G/C_G(V_i)$ is nonabelian that

$$|G : G'| = |G/C_G(V_i) : (G/C_G(V_i))'| |C_G(V_i) : C_G(V_i)'|$$

and $|G/C_G(V_i) : (G/C_G(V_i))'|$ is the largest orbit size of G on V_i . Again by induction, we have that $G/C_G(V_i) = K_i \times L_i$, where $K_i \in \text{Syl}_2(G/C_G(V_1))$ is nonabelian and L_i is an abelian group of odd order. Since G is nilpotent and is isomorphic to a subgroup of $\bigtimes_{i=1}^n G/C_G(V_i)$, we get the desired conclusion that the Hall 2'-subgroup of G is abelian. From now on, we may assume that V is irreducible.

Step 2. A reduction to the case that G is a p -group

Suppose that G is not a p -group for any prime p . Let p be a prime dividing $|G|$ such that if P is a Sylow p -subgroup of G , then P is nonabelian. Then $G = P \times H$, where

H is the Hall p' -subgroup of G . Note that $H > 1$. By Step 1, V is a G -module over a field K . By Lemma 10 of [19], there exists a field extension L of K such that if U is an irreducible summand of V viewed as an LG -module, then U is absolutely irreducible, and the permutation actions of G on V and U are permutation isomorphic. This allows us to consider the action of G on U instead of V ; in other words, we may assume that V is absolutely irreducible. Using (3.16) in [1], we have that $V = X_1 \otimes X_2$, where X_1 is a faithful irreducible P -module and X_2 is a faithful irreducible H -module. Let us pick an $x_1 \in X_1$ and a $x_2 \in X_2$ such that $|x_1^P|$ is the largest orbit size in the action of P on X_1 and $|x_2^H|$ is the largest orbit size in the action of H on X_2 . It follows from Theorem 2.1 that $|P/P'| \leq |x_1^P|$ and $|H/H'| \leq |x_2^H|$. If $g \in P$ and $h \in H$ satisfying $gh \in C_G(x_1 \otimes x_2)$, then we have by Lemma 3.3 of [14] that $x_1g = \alpha x_1$ and $x_2h = \beta x_2$, where α, β are scalars in the field with $\alpha\beta = 1$. Now g and h have coprime orders, and hence we have that $\alpha = \beta = 1$, which gives us $C_G(x_1 \otimes x_2) = C_P(x_1) \times C_H(x_2)$. Now we obtain the following inequality that

$$\begin{aligned} |G : G'| = M &\geq |(x_1 \otimes x_2)^G| = |G : (C_P(x_1) \times C_H(x_2))| \\ &= |P : C_P(x_1)| |H : C_H(x_2)| \\ &= |x_1^P| |x_2^H| \geq |P/P'| |H/H'| = |G : G'|. \end{aligned}$$

Since we have equality everywhere in the above inequality, we conclude that $|x_1^P| = |P/P'|$ and $|x_2^H| = |H/H'|$. Therefore, $|P/P'|$ is the largest orbit size of P on X_1 and similarly $|H/H'|$ is the largest orbit size of H on X_2 . Since P is nonabelian, by induction applied to the action of P on X_1 we obtain that $p = 2$. Hence $|H|$ is odd, and so by induction applied to the action of H on X_2 we get that H is abelian, as desired.

From now on, we may assume that G is a p -group for some prime p , and we wish to show $p = 2$.

Step 3. Final conclusion.

By Lemma 2.5, we may assume that V is not quasiprimitive. Thus, it is well-known that there exists a normal subgroup D of G such that $|G : D| = p$ and $V_D = V_1 \oplus \dots \oplus V_p$ for irreducible D -submodules V_i of V_D . If D is abelian, then D has a regular orbit on V and hence

$$M \geq |D| = \frac{p|D|}{p} = \frac{|G|}{p} \geq \frac{|G|}{|G'|} = M.$$

Therefore, we get $|D| = |G : G'|$, and hence we find that $|G'| = p$. It follows from Theorem 2.4 that $p = 2$, as desired. Thus, in the rest of the proof we may assume that D is a nonabelian group.

If $C_D(V_i) = 1$ for some $i \in \{1, 2, \dots, p\}$, then D acts faithfully on V_i . Without loss of generality suppose that $C_G(V_1) = 1$. We know from Theorem 2.1 that $|D : D'| \leq M_1$, where M_1 is the largest orbit size of D on V_1 . If $|D : D'| = M_1$, then by induction we find $p = 2$. Let $|D : D'| < M_1 \leq M$. Then we have $M = |G : G'| \leq |G : D'| = p|D :$

$|D'| \leq M_1 \leq M$, and hence we find $M = M_1$. Let $v_1 \in V_1$ be a representative of an orbit of size M_1 of D on V_1 . Then v_1^D is an orbit of D of size $M_1 = M$, so it is also a maximal orbit of G on V . However, v_1^D is a subset of V_1 , whereas v_1^G is not, since the elements in $G - D$ permute as p -cycles the elements of the set $\{V_1, \dots, V_p\}$. So v_1^D is a proper subset of v_1^G , which implies $|v_1^G| > M$, contradiction. Thus, for the rest of the proof we may assume that $C_D(V_i) > 1$ for all $i = 1, 2, \dots, p$. Now we examine two cases, where $D' < G'$ and $D' = G'$.

Step 3.1. The case where $D' < G'$

It follows that $p|D'| \leq G'$, and hence

$$M \geq |D : D'| = \frac{p|D|}{p|D'|} \geq \frac{|G|}{|G'|} = M.$$

Therefore, we get $|D : D'| = M$ and observe that M is the maximal orbit size in the action of D on V_D . By induction, we find that $p = 2$, which completes this case where $D' < G'$.

Step 3.2. The case where $D' = G'$

We have that D is isomorphic to a subgroup of $\bigtimes_{i=1}^p D/C_D(V_i)$. Since G permutes the V_i transitively, if $D/C_D(V_i)$ is abelian for any $i = 1, \dots, p$, then all $D/C_D(V_i)$ must be abelian. Thus, D becomes abelian, which is a contradiction. Hence all $D/C_D(V_i)$ must be nonabelian for $i = 1, 2, \dots, p$. Let M_1 be the largest orbit size in the action of D on V_1 and let M_2 be the largest orbit size of $C_D(V_1)$ on $V_2 \oplus \dots \oplus V_p$. To simplify the notation, throughout this proof let W denote $V_2 \oplus \dots \oplus V_p$. Let $x \in V_D$ be in a largest orbit of D on V_D . Then by Lemma 2.7 (a) we have that $M \geq M_D = M_1 M_2 \geq |D : D'|$. We next consider the following inequalities (which we get from Theorem 2.1):

$$|D : D'C_D(V_1)| \leq M_1 \tag{3}$$

$$|C_D(V_1) : C_D(V_1)'| \leq M_2 \tag{4}$$

If Inequalities (3) and (4) are strict, then we have that $p|D : D'C_D(V_1)| \leq M_1$ and $p|C_D(V_1) : C_D(V_1)'| \leq M_2$. Therefore, we have by Lemma 2.2 that

$$\begin{aligned} M &\geq M_1 M_2 = M_D \geq p^2 |D : D'C_D(V_1)| |C_D(V_1) : C_D(V_1)'| \\ &\geq p^2 |D : D'C_D(V_1)| |C_D(V_1) : D' \cap C_D(V_1)| \\ &\geq p^2 |D : D'| = p|G : D| |D : D'| = p|G : D'| = p|G : G'| \\ &> |G : G'| = M, \end{aligned}$$

which is a contradiction. If we have equality in (3), considering the fact that D is non-abelian and using the inductive hypothesis, we find the desired conclusion that $p = 2$.

So it remains to consider the case where we have strict inequality in (3) and equality in (4). Thus, we have $p|D : D'C_D(V_1)| \leq M_1$, and hence by using Lemma 2.2,

$$\begin{aligned}
M \geq M_1 M_2 = M_D &\geq p|D : D' C_D(V_1)| |C_D(V_1) : C_D(V_1)'| \\
&\geq p|D : D' C_D(V_1)| |C_D(V_1) : D' \cap C_D(V_1)| \\
&\geq p|D : D'| = |G : G'| = M.
\end{aligned}$$

Therefore, we have equality everywhere which gives us $M = M_D = M_1 M_2$, $M_1 = p|D : D' C_D(V_1)|$ and $C_D(V_1)' = D' \cap C_D(V_1)$.

Set $C_i := C_D(V_i)$ for all $i = 1, \dots, p$. If C_1 is nonabelian, then by applying induction to the nonabelian group C_1 acting on W , we get the desired result that $p = 2$. From now on, we may assume that the C_i are abelian for all $i = 1, \dots, p$. Therefore, we have $C_i \cap D' = C'_i = 1$ for all i , which leads to $C_i \leq Z(D)$ for all i . Let us define $B_i = C_D(V_1 \oplus \dots \oplus V_{i-1} \oplus V_{i+1} \oplus \dots \oplus V_p)$ for $i = 1, \dots, p$. It should be noted that the B_i are all conjugate to each other in G . We claim that $B_i > 1$ for all i . To see this, it suffices to show that $B_p > 1$. Working towards a contradiction, assume that $B_p = 1$. Then $D \cong D/B_p$ acts faithfully on $V_1 \oplus \dots \oplus V_{p-1}$, and hence C_1 acts faithfully on $V_2 \oplus \dots \oplus V_{p-1}$. Since C_1 is abelian, it has a regular orbit on $V_2 \oplus \dots \oplus V_{p-1}$. (Note that even if $p = 2$, the trivial group $B_p = C_1 = 1$ has regular orbit on the trivial module $V_2 \oplus \dots \oplus V_{p-1} = 0$.) Let $w \in V_2 \oplus \dots \oplus V_{p-1}$ be a representative of such a regular orbit of C_1 on $V_2 \oplus \dots \oplus V_{p-1}$, and let $v_1 \in V_1$ be a representative of an orbit of size M_1 of D on V_1 . Then $v_1 + w$ is a representative of an orbit of size $M_1 M_2 = M$ of D on V . However $(v_1 + w)^D$ is a subset of $V_1 \oplus \dots \oplus V_{p-1}$, whereas $(v_1 + w)^G$ is not, since elements in $G - D$ permute as p -cycles the elements of the set $\{V_1, \dots, V_p\}$. Thus, $(v_1 + w)^D$ is a proper subset of $(v_1 + w)^G$, in other words, $|(v_1 + w)^G| > |(v_1 + w)^D|$. This contradicts the fact that M is the largest orbit size of G on V , and hence this proves the claim that $B_i > 1$ for all i .

Since $B_i \leq C_j \leq Z(D)$ for all $j \neq i$, we see that $B_i \leq Z(D)$ for all i . Since B_i acts faithfully on V_i and trivially on V_j for all $i \neq j$, we conclude that $B_i \cap \prod_{j \neq i} B_j = 1$ for all i . Hence if we define $B = B_1 B_2 \dots B_p$, then we see $B = B_1 \times B_2 \times \dots \times B_p$. Note that B is a normal subgroup of G because every element in $G - D$ permutes the B_i in a p -cycle.

Let us pick $g \in G - D$ and observe that g^p acts trivially on B . Hence we can consider the semidirect product H of B and $\langle g \rangle / \langle g^p \rangle$ with respect to the action of $\langle g \rangle / \langle g^p \rangle$ on B induced by the action of g on B . Thus $H \cong B_1 \wr (\langle g \rangle / \langle g^p \rangle)$. Also, as $B \leq Z(D)$, we see that $[[B, \langle g \rangle]] = [[B, \langle g \rangle / \langle g^p \rangle]]$. Thus by Lemma 2 of [8], we have that

$$|[B, \langle g \rangle]| = |[B, \langle g \rangle / \langle g^p \rangle]| = |B_1|^{p-1}.$$

Now let $N = B_2 \times \dots \times B_p \leq C_1$ and note that $D' \cap N \leq D' \cap C_1 = 1$. Then we have

$$\frac{|D'| |B|}{|D' \cap B|} = |D' B| = |D' N B_1| = \frac{|D' N| |B_1|}{|D' N \cap B_1|} = \frac{|D'| |N| |B_1|}{|D' N \cap B_1|} = \frac{|D'| |B|}{|D' N \cap B_1|}.$$

Thus, we deduce that $|D' \cap B| = |D' N \cap B_1| \leq |B_1|$. Since $[B, \langle g \rangle] \leq B \cap G' = B \cap D'$, we have

$$|B_1|^{p-1} = |[B, \langle g \rangle]| \leq |D' \cap B| \leq |B_1|.$$

As $|B_1| > 1$, we then get $p - 1 \leq 1$, which gives us $p = 2$. This completes the proof of the theorem. $\square$

We are ready to give the proof of Theorem A.

Proof of Theorem A. Put $M = |G : G'|$. To prove the theorem, we shall argue by induction on $|GV|$. We will first show that $G' = \Phi(G)$ is of order 2 by using induction. We know from Theorem 3.1 that $G = P \times A$, where P is the Sylow 2-subgroup of G and A is an abelian group of odd order. Since G is a nonabelian group, we have $P > 1$. On the other hand, since G acts irreducibly on V , the center $Z(G)$ is cyclic, and hence A is a cyclic group.

Step 1. A reduction to the case that G is a 2-group

Suppose that G is not a 2-group, that is, $A > 1$. By the same arguments as in Step 2 of the proof of Theorem 3.1, we may assume that $V = X_1 \otimes X_2$, where X_1 is a faithful irreducible P -module and X_2 is a faithful irreducible A -module, and hence $|P/P'|$ becomes the largest orbit size in the action of P on X_1 . We can use induction to tell us that P acting on X_1 is one of the group actions listed in the theorem. Suppose that $n := \dim(X_2) > 1$. Then we have that $V_P = W \oplus \dots \oplus W =: nW$, where $Y \cong X_1$ (as P -modules). Likewise, $V_A = Y \oplus \dots \oplus Y =: mY$, where $m = \dim(X_1) \in \{2, 4\}$, and $Y \cong X_2$ (as A -modules). Now it is easy to see that P has a regular orbit on $X_1 \oplus X_1$ (e.g. by Theorem 4.7 of [15]). As $n > 1$, it follows that P has a regular orbit on V . Since the cyclic group A acts Frobeniusly on V and P has a regular orbit on V , we conclude that $G = P \times A$ has a regular orbit on V , contradiction. Now let us assume that $n = 1$. If X_1 is as in Case (a) or (b) of the theorem, then X_2 is $V(1, 3)$ and $V(1, 5)$, respectively. Therefore 2 divides $|A|$, which is a contradiction. If $X_1 \cong V(4, 3)$ is as in Case (c), then $X_2 = V(1, 3)$ or $V(1, 9)$. This gives us again the same contradiction that 2 divides $|A|$. Thus from now on we shall assume that G is a 2-group.

Step 2. G' is of order 2.

Since V is an imprimitive irreducible G -module from Lemma 2.5, there exists $D \trianglelefteq G$ satisfying $|G : D| = 2$ and $V_D = V_1 \oplus V_2$ for irreducible D -submodules V_i of V_D for $i \in \{1, 2\}$. Suppose that D is abelian. Then D has a regular orbit on V , and thus

$$M \geq |D| = \frac{2|D|}{2} = \frac{|G|}{2} \geq \frac{|G|}{|G'|} = M.$$

Therefore, we get that $|D| = |G : G'|$, and hence $|G'| = 2$, and we are done. Let us assume that D is nonabelian.

Now let us suppose that $C_D(V_1) = 1$, thus also $C_D(V_2) = 1$. Then D acts faithfully on V_1 , and from Theorem 2.1 we have that $|D : D'| \leq M_1 \leq M$, where M_1 is the largest orbit size of D on V_1 . If $M_1 = M$ and $v_1 \in V_1$ is a representative of an orbit of size M_1

of D on V_1 , then v_1^D is an orbit of D of size $M_1 = M$, so it is also a maximal orbit of G on V . However, v_1^D is a subset of V_1 , whereas v_1^G is not a subset of V_1 . It follows that v_1^D is a proper subset of v_1^G , which implies $|v_1^G| > M$, which is a contradiction. So let us assume that $M_1 < M$. Thus, obtain the inequality

$$M \geq 2M_1 \geq 2|D : D'| \geq |G : G'| = M,$$

which shows that $|D : D'| = M_1$ and $D' = G'$. By induction, we obtain $2 = |D'|$, and hence we get the desired result $2 = |G'|$. From now on, we shall assume that $C_D(V_i) > 1$ for $i = 1, 2$. The remainder of our proof will require several technical cases and analysis of the action of D on V_D . Let us now consider two cases, where $D' < G'$ and $D' = G'$.

Step 2.1. The case where $D' < G'$

We have the inequality $2|D'| \leq G'$, which implies that

$$M \geq |D : D'| = \frac{|G|}{2|D'|} \geq \frac{|G|}{|G'|} = M.$$

Hence $|D : D'| = M$, and M is the maximal orbit size in the action of D on V_D . Considering the fact that the nonabelian group D is isomorphic to a subgroup of $D/C_D(V_1) \times D/C_D(V_2)$, the factor groups $D/C_D(V_i)$ cannot be abelian for $i = 1, 2$. To simplify the notation for the rest of the proof, let us write $C_i = C_D(V_i)$ for $i = 1, 2$, M_1 for the maximal orbit size in the action of D/C_1 on V_1 , and M_2 for the maximal orbit size in the action of C_1 on V_2 . Now applying Lemma 2.3, we have that

$$M = |D : D'| = |D/C_1 : (D/C_1)'| |C_1 : C'_1|,$$

where $|D/C_1 : (D/C_1)'|$ is the maximal orbit size in the action of D/C_1 on V_1 and $|C_1 : C'_1|$ is the maximal orbit size in the action of C_1 on V_2 , and Lemma 2.2 yields that $C_1 \cap D' = C'_1$. Thus, we can use induction to tell us that $|(D/C_1)'| = |(D/C_2)'| = 2$. Since D is isomorphic to a subgroup of $D/C_1 \times D/C_2$, we have that $|D'| \in \{2, 4\}$.

First suppose that $|D'| = 2$. Since $(D/C_i)' \cong D'/(D' \cap C_i)$ is of order 2 for all $i = 1, 2$, we have that $C_1 \cap D' = C'_1 = 1$ and $C_2 \cap D' = C'_2 = 1$. Thus, we conclude that $C_i \leq Z(D)$ for $i = 1, 2$. Let us pick $v_1 \in V_1$ such that v_1^{D/C_1} is an orbit of maximal size. Hence we have

$$|D : D'C_1| = |D/C_1 : (D/C_1)'| = |v_1^D| = |D : C_D(v_1)|,$$

which shows that $|C_D(v_1)| = |D'C_1| = 2|C_1|$. Since $C_1 \leq Z(D)$, we conclude that $C_D(v_1)$ is abelian. Next we shall need that the action of $C_D(v_1)$ is faithful on V_2 . To show this, assume that this action of $C_D(v_1)$ on V_2 is not faithful. Then there is an element $1 \neq x \in C_D(v_1)$, which acts trivially on V_2 , that is, $x \in C_D(v_1) \cap C_2$. Since we already know that $C_2 \leq Z(D)$, we get $x \in Z(D)$. Let us consider the set $C_{V_1}(x)$ and note that $0 \neq v_1 \in C_{V_1}(x)$. Since $x \in Z(D)$, clearly $C_{V_1}(x)$ is a D -module. Since V_1 is an

irreducible D -module and $C_{V_1}(x) > 0$, we conclude that $C_{V_1}(x) = V_1$. So $x \in C_1$, and altogether we have $x \in C_1 \cap C_2 = 1$, contradicting $x \neq 1$. Thus, we deduce that $C_D(v_1)$ acts faithfully on V_2 . Then the abelian group $C_D(v_1)$ has a regular orbit $v_2^{C_D(v_1)}$ on V_2 for some $v_2 \in V_2$. Since

$$|D : C_D(v_1 + v_2)| = |D : C_D(v_1) \cap C_D(v_2)| = |D : C_D(v_1)| |v_2^{C_D(v_1)}| = |D|,$$

we deduce that $v_1 + v_2$ is in a regular orbit of D on V_D . Since M is the maximal orbit size of D on V_D , we have $|D| \leq M = |D : D'|$, which leads a contradiction that $D' = 1$.

Now let us assume that $|D'| = 4$, and hence $|G'| = 8$. We also have $|C'_1| = |C'_2| = 2$ because $(D/C_i)' \cong D'/(D' \cap C_i)$ are of order 2 and $D' \cap C_i = C'_i$ for $i = 1, 2$. Since C_1 is nonabelian, we have $|C_1| \geq 8$. And since $C'_1 \leq Z(D)$, we know that C_1/C'_1 is not cyclic. Hence if we put $H = \langle g, C_1 C_2 \rangle \leq G$ for some $g \in G - D$ and $S = C'_1 \times C'_2$, then there exist some elements x and y in C_1 such that xS and yS are both of order 2 in H/S , they commute with each other, and $xS \neq yS$. In other words, $\langle xS, yS \rangle$ is a Klein 4-group. Clearly, we have that $C_1^g = C_2$ and $C_2^g = C_1$ because $g \in G - D$ and G permutes V_1 and V_2 . Hence $(xS)^g \in C_2 S/S$ and $(yS)^g \in C_2 S/S$. Therefore, the group

$$T := \langle (x^{-1}S)(xS)^{(gS)}, (y^{-1}S)(yS)^{(gS)} \rangle = \langle (xS)(xS)^{(gS)}, (yS)(yS)^{(gS)} \rangle$$

becomes a Klein 4-subgroup of H/S . On the other hand, observe that $T = \langle [xS, gS], [yS, gS] \rangle$, which shows that T is a subgroup of $(H/S)'$. So $(H/S)'$ has a subgroup T of order 4, and the inverse image T_0 of that subgroup in H is a subgroup of H' because $S \leq H'$. This yields $|T_0| = 16$, which leads a contradiction that $16 \leq |H'| \leq |G'| = 8$. This final contradiction completes the case where $D' < G'$.

Step 2.2. The case where $D' = G'$

Recall that since D is nonabelian, both D/C_1 and D/C_2 are nonabelian. By using Lemma 2.7 (a), we find that $M \geq M_D = M_1 M_2$, where M_D is the maximal orbit size in the action of D on V_D . From now on, we consider the inequalities $|D : D' C_1| \leq M_1$ and $|C_1 : C'_1| \leq M_2$. First suppose that $|D : D' C_1| < M_1$ and $|C_1 : C'_1| < M_2$. Then we have $2|D : D' C_1| \leq M_1$ and $2|C_1 : C'_1| \leq M_2$. It follows that

$$\begin{aligned} M &\geq M_D = M_1 M_2 \geq 4|D : D' C_1| |C_1 : C'_1| \\ &\geq 4|D : D'| = 2|G : G'| > |G : G'| = M, \end{aligned} \tag{5}$$

which is a contradiction.

Let us assume that $|D : D' C_1| = M_1$ and $|C_1 : C'_1| < M_2$. By using the fact that $2|C_1 : C'_1| \leq M_2$, we have

$$\begin{aligned} M &\geq M_1 M_2 = M_D \geq 2|D : D' C_1| |C_1 : C'_1| \\ &\geq 2|D : D' C_1| |C_1 : D' \cap C_1| \geq 2|D : D'| = |G : G'| = M, \end{aligned} \tag{6}$$

which gives us equality everywhere, and hence $C'_1 = D' \cap C_1$. By induction we have $|(D/C_1)'| = 2$, and similarly $|(D/C_2)'| = 2$. Observing the fact that D is isomorphic to a subgroup of $D/C_1 \times D/C_2$, we find that $D' \in \{2, 4\}$. If $|D'| = 2$, then we are done due to $D' = G'$. Let us suppose, by way of contradiction, that $4 = |D'| = |G'|$. Thus, we obtain $|C'_1| = |C'_2| = 2$ since $(D/C_i)' \cong D'/(D' \cap C_i)$ is of order 2 and $D' \cap C_i = C'_i$ for $i = 1, 2$. And since $C'_1 \leq Z(D)$, we know that C_1/C'_1 is not cyclic. Let $g \in G - D$ and let us define the groups H and S as in the case (b) of Lemma 2.7. By Inequality (1), we have that $S < H'$, which is a contradiction because $H' \leq G'$ and $|G'| = |S| = 4$.

We next consider the case where $|D : D'C_1| = M_1$ and $|C_1 : C'_1| = M_2$. Thus, we have that

$$\begin{aligned} 2|D : D'| = M &\geq M_D = M_1M_2 = |D/C_1 : (D/C_1)'||C_1 : C'_1| \\ &\geq |D/C_1 : (D/C_1)'||C_1 : C_1 \cap D'| = |D : D'|. \end{aligned} \quad (7)$$

By induction, we know that $|(D/C_1)'| = |(D/C_2)'| = 2$, and hence as before we obtain that $|D'| \in \{2, 4\}$ because D is isomorphic to a subgroup of $D/C_1 \times D/C_2$. If $|D'| = 2$, then we get the desired result that $|G'| = 2$. To find a contradiction, let us assume that $|D'| = 4$. If C_1 is nonabelian, then $1 < C'_1 \leq D' \cap C_1$. Since $2 = |(D/C_i)'| = |D'/D' \cap C_i|$, we get $|C'_i| = 2$ for $i = 1, 2$. If we define the groups H and S as in the case (b) of Lemma 2.7 and by using (1), then we get a contradiction that $S < H' \leq G'$ because $|G'| = |S| = 4$.

Now suppose that C_1 is an abelian group. Then we again have $2 = |D' \cap C_i|$ for $i = 1, 2$. Notice that $D'C_i$ is normal in G for $i = 1, 2$ because we are in the case that $D' = G'$. Also, since we have $C_1^g = C_2$ for $g \in G - D$, we obtain $D'C_1 = D'C_2$, which shows $C_1C_2 \leq D'C_1$. In fact, $C_1C_2 = D'C_1$ because $|D'C_1/C_1| = 2$. Considering the fact that $|D'| = 4$ and $|D' \cap C_1| = 2$, we get

$$|C_1|^2 = |C_1C_2| = |D'C_1| = |D'| |C_1| / |D' \cap C_1| = 2|C_1|,$$

and thus we find that $|C_1| = |C_2| = 2$. Therefore, $D' = C_1 \times C_2 \leq Z(D)$, and also $M = M_D = M_1M_2$ by using (7). Let us pick $v_1 \in V_1$ as a representative of an orbit of size M_1 of D on V_1 . Then $|D : C_D(v_1)| = M_1 = |D : D'C_1|$, and so $|C_D(v_1)| = |D'C_1| = 4|C_1|/2 = 2|C_1| = 4$. So $C_D(v_1)$ is abelian of order 4 and clearly $C_D(v_1) \not\leq Z(D)$ because of $C_V(Z(D)) = 1$. Since we have $C_1 \leq C_D(v_1)$ and $D' \leq Z(D)$, this shows that $C_D(v_1) \cap C_2 = 1$. As a consequence of this, we conclude that $C_D(v_1)$ acts faithfully on V_2 . It follows that the abelian group $C_D(v_1)$ has a regular orbit on V_2 . If $v_2 \in V_2$ is a representative of such a regular orbit, we easily see that

$$M = M_D = |(v_1 + v_2)^D| \geq |v_1^D||v_2^{C_D(v_1)}| = M_14 = |D : D'|4 = 2M,$$

which is a contradiction.

We are finally left with the case that $|D : D'C_1| < M_1$ and $|C_1 : C'_1| = M_2$. Repeating the same process in (6), we find that $M_1 = 2|D : D'C_1|$, $M = M_1M_2 = M_D$ and

$C'_1 = D' \cap C_1$. First assume that D/C_1 has a regular orbit on V_1 . Then D/C_2 has a regular orbit on V_2 , and hence we deduce that $C_1C_2/C_2 \cong C_1$ has a regular orbit on V_2 . Thus, we have $|C_1| = M_2 = |C_1 : C'_1|$, showing that $C'_1 = 1$, and hence C_1 is abelian. Moreover,

$$|D/C_1| = M_1 = 2|D : D'C_1| = |D/C_1| \frac{2|C'_1|}{|D'|} = |D/C_1| \frac{2}{|D'|},$$

which implies $|D'| = 2$. Since $G' = D'$, we get $|G'| = 2$, which is the desired conclusion.

So now we may assume that D/C_1 has no regular orbit on V_1 . Let $v_1 \in V_1$ be a representative of an orbit of D of size M_1 . Then $C_D(v_1) > C_1$ (*) because $1 < C_{D/C_1}(v_1) = C_D(v_1)/C_1$. Let $d \in D$ and $c \in C_1$. Then we get that $[dC'_1, cC'_1] = [d, c]C'_1 = C'_1$ since $[d, c] \in D' \cap C_1 = C'_1$. It follows that $C_1/C'_1 \leq Z(D/C'_1)$. Next we observe that $|D : C_D(v_1)| = M_1 = 2|D : D'C_1|$ from which we derive that

$$|C_D(v_1)| = |D'C_1|/2 = \frac{|D'| |C_1|}{2|C'_1|}. \quad (8)$$

Since $D'C_1 = G'C_1$ is a normal subgroup of G and G permutes C_1 and C_2 , we conclude that $G'C_1 = D'C_1 = D'C_2$. Hence $D'C_1 = D'C_2$ contains the group $C_1C_2 = C_1 \times C_2$. Let $S := C'_1 \times C'_2$. Then by Lemma 2.7 (b), we see that

$$|C_1/C'_1| \leq |[g, (C_1 \times C_2)/S]|.$$

Hence, we conclude that $|C_1/C'_1| \leq |(G/S)'| = |G'/S| = |D'/S|$, which yields the inequality that

$$|D'| \geq |C_1/C'_1| |S| = |C_1/C'_1| |C'_1|^2 = |C_1| |C'_1|.$$

It immediately shows by Equation (8) that $|C_D(v_1)| \geq |C_1|^2/2$.

Next we claim that $C_D(v_1)$ is not a subgroup of C_1C_2 . To see this, suppose that $C_D(v_1) \leq C_1C_2$. Let us first assume that C_1 is abelian. Since $D' \cap C_i = C'_i = 1$ for $i = 1, 2$, we have that $C_1C_2 \leq Z(D)$. Observe that $C_2 \cong C_1C_2/C_1$ is a subgroup of the cyclic group $Z(D/C_1)$. This shows that the cyclic group C_2 acts Frobeniusly on V_1 , which implies that $C_{C_1C_2}(v_1) = C_1$. Recall from (*) that $C_D(v_1) > C_1$, so that altogether it follows $C_D(v_1) \not\leq C_1C_2$ in this case. This leaves us with the case that C_1 is nonabelian. Then C_2 is also nonabelian. Since $1 < C'_2 \cap Z(D) \cong (C'_2 \cap Z(D))C_1/C_1 \leq Z(D/C_1)$ and $Z(D/C_1)$ is cyclic, we conclude that the action of $C'_2 \cap Z(D)$ on V_1 is Frobenius. Now $C'_2 \cap Z(D)$ is a subgroup of the Frattini subgroup $\Phi(C_2)$ such that no nontrivial element of $C'_2 \cap Z(D)$ centralizes v_1 , and hence some elements in $C_2 - \Phi(C_2)$ cannot centralize v_1 . Thus $|C_2 : C_{C_2}(v_1)| \geq 2$. If $|C_2 : C_{C_2}(v_1)| = 2$, then $C'_2 \cap Z(D) \leq \Phi(C_2) \leq C_{C_2}(v_1)$, which is a contradiction because the action of $C'_2 \cap Z(D)$ on V_1 is Frobenius. It follows that $|C_2 : C_{C_2}(v_1)| \geq 4$, which implies that $|C_D(v_1)| \leq |C_2|/4$. This proves

that $|C_{C_1C_2}(v_1)| \leq |C_1C_2|/4 = |C_1|^2/4$. Together with $|C_D(v_1)| \geq |C_1|^2/2$ from above this forces that $|C_{C_1C_2}(v_1)| < |C_D(v_1)|$. That is, $C_D(v_1)$ cannot be a subgroup of C_1C_2 , completing the proof of our claim.

So let $x \in C_D(v_1) - C_1C_2$ and put $K = \langle x \rangle C_1$. Since $C_1/C'_1 \leq Z(D/C'_1)$, we see that K/C'_1 is abelian, and thus $K' = C'_1$. Let L be the kernel of the action of K on V_2 . Then $L \cap C_1 = 1$ because $L = C_K(V_2) \leq C_2$. Thus $L \cong LC_1/C_1 \leq \langle x \rangle C_1/C_1 = K/C_1$, showing that L is cyclic, and $L \leq C_2$. Since $x \notin C_1C_2$ and $L \leq C_2$, we obtain $x \notin LC_1$. Hence even $LC_1/C_1 < K/C_1$ and $LC_1 < K$. This shows that K has the following properties:

- i. K/L acts faithfully and completely reducibly on V_2 ;
- ii. $(K/L)' = K'L/L = C'_1L/L \cong C'_1$;
- iii. $|(K/L)/(K/L)'| = \frac{|K|}{|L||C'_1|} \geq \frac{2|LC_1|}{|L||C'_1|} = \frac{2|L||C_1|}{|L||C'_1|} = \frac{2|C_1|}{|C'_1|} = 2|C_1 : C'_1| = 2M_2$;
- iv. $K \leq C_D(v_1)$.

Now by (i), (iii) and Theorem 2.1, we know that K/L has an orbit of size at least $2M_2$ on V_2 . Also, by (iv) we have that $C_D(v_2)$ has an orbit of size at least $2M_2$ on V_2 , so let $v_2 \in V_2$ be a representative of such an orbit. With this we get

$$M = M_D \geq |(v_1 + v_2)^D| \geq |v_1^D||v_2^{C_D(v_1)}| \geq M_1 2M_2 = 2M_1 M_2 = 2M,$$

which is a contradiction. This completes the proof of Step 2 and from now on we have $|G'| = 2$.

Step 3. The Frattini Subgroup $\Phi(G)$ of G is of order 2.

We now restrict our attention to the special cases that D is abelian and nonabelian.

Step 3.1. D is an abelian group.

By Lemma 2.7 (a), we know that $M \geq M_1 M_2 = M_D \geq |D : D'| = |D|$. If $C_1 = 1$, then the action of D is faithful on V_1 , and so D has a regular orbit on V_1 , say v_1^D . Then for $g \in G - D$, we know that $v_1^G = (v_1^D)^g \notin V_1$, and hence we get a contradiction that $|v_1^G| > |v_1^D| = M_1 = M$ because we have $M = |G : G'| = |D|$. Therefore, we conclude that if D is abelian, then $|C_1| = |C_2| > 1$.

Since G/D is cyclic and D is a maximal abelian subgroup of G , we find by Lemma 12.12 of [6] that $|D| = |D \cap Z(G)||G'| = |Z(G)||G'|$ because $Z(G) \leq C_G(D) = D$. Thus, we conclude that $|D : Z(G)| = 2$, which means $|G : Z(G)| = 4$. In fact, since $Z(G)$ acts Frobeniusly on V_1 , we know that $C_1 \cap Z(G) = 1$, that is, $D = C_1 \times Z(G)$ because $|D : Z(G)| = 2$. It follows that $2 = |C_1| = |C_2|$. Since the action of $Z(G)$ on V is Frobenius, we have $C_G(v) \cap Z(G) = 1$, and hence, as G is nonabelian, $C_G(v) \times Z(G) < G$ for all $0 \neq v \in V$. This gives us $|C_G(v)| = 2$ for all nonzero vectors v in V because $|G : Z(G)| = 4$. (We pause here to remark that at this point, to finish this case we could use the complete classification of imprimitive linear group actions with all orbits having the same size and not being regular, as provided for the solvable case by [17, Proposition 1.3], but we prefer to give an independent argument here that is more in the spirit of the

arguments used throughout this paper.) Moreover, $\Phi(D) = \Phi(Z(G)) \leq \Phi(G) \leq Z(G)$ and observing that $Z(G)$ is a cyclic group and $|G : Z(G)| = 4$, we conclude $|G : \Phi(G)| = 8$. Since we have $|C_G(v)| = 2$ for all nonzero vectors v in V and $C_G(V) = 1$, we have at least two different elements a and b in $G - Z(G)$ such that $\circ(a) = \circ(b) = 2$. Observe that a and b are not in $\Phi(G)$ because $\Phi(G) \leq Z(G)$. Thus, there exists $z \in Z(G)$ such that $G/\Phi(G) = \langle a\Phi(G), b\Phi(G), z\Phi(G) \rangle$. This leads to $G = \langle a, b, c \rangle$ since $\Phi(G)$ is equal to the set of non-generators of G .

Next we will show that $U := \langle a, b \rangle \cong D_8$. It is well known that a group generated by two involutions is a dihedral group and hence $|Z(U)| = 2$. Suppose that $U = \langle a, b \rangle \cong D_{2^n}$ and $n > 3$. Clearly $U/\Phi(U) = \langle a\Phi(U), b\Phi(U) \rangle$. Let $x \in U$ be of order 8. Then $x\Phi(U) \in U/\Phi(U)$ and $x^2 \in \Phi(U) \leq \Phi(G) \leq Z(G)$. It follows that $x^2 \in Z(U)$, but as x^2 is of order 4, this is a contradiction to $|Z(U)| = 2$. Thus, we get $n = 3$ and the desired conclusion that $U = \langle a, b \rangle \cong D_8$. It follows that $G = \langle a, b, c \rangle \cong D_8 \circ Z(G)$. Now if $|\Phi(G)| \geq 4$, then by Lemma 2.6, G has a regular orbit on V , which is a contradiction. This shows $|\Phi(G)| = 2$, as desired. Thus for the rest of the proof, we shall assume that D is a nonabelian group.

Step 3.2. D is a nonabelian group.

Observe that $D' = G'$ is of order 2. We next show that $C_i > 1$ for $i = 1, 2$. To do this, let us assume that $C_1 = C_2 = 1$. Since D acts faithfully and irreducibly on V_1 , we conclude that $Z(D)$ is cyclic and D' becomes a unique minimal normal subgroup of D . By Theorem 2.1, we know that $|D : D'| \leq M_1 \leq M$. If $M_1 = M$, then $|v_1^D| = M$ for some $v_1 \in V_1$, and hence v_1 is also in a maximal orbit of G on V . This gives us a contradiction $|v_1^G| > |v_1^D| = M$ because $v_1^D \subseteq V_1$ whereas $v_1^G \not\subseteq V_1$. It follows that $|D : D'| \leq M_1 < M$, and hence we get

$$M \geq 2M_1 \geq 2|D : D'| \geq |G : G'| = M.$$

This leads to $|D : D'| = M_1$. By induction, we obtain that $D' = \Phi(D)$ is of order 2. Since $D/\Phi(D)$ is an elementary abelian group, it follows that $Z(D)$ is cyclic of order 2 or 4, and as $\Phi(D) \leq Z(G) \leq Z(D)$, we conclude that $Z(G) = \Phi(D) = D'$ or $\Phi(G) < Z(G) = Z(D)$. Since G' is the unique minimal normal subgroup of G , we have by Lemma 12.3 of [6] that $G/Z(G)$ is an elementary abelian group and $|G : Z(G)| = \chi(1)^2$ for all nonlinear irreducible characters χ of G ; in particular, $\Phi(G) \leq Z(G)$. Thus, if $Z(G) = \Phi(D) = D'$, then we get $|\Phi(G)| = 2$, as desired. Let us assume that $\Phi(G) < Z(G) = Z(D)$. We observe that D (still by induction) is the group as in the case (b) of the theorem, and so we have $|D : Z(D)| = 4$. By considering the fact that $\chi(1)^2 = |G : Z(G)| = 2|D : Z(D)| = 8$, we get a contradiction. From now on, we shall assume that $C_i > 1$ for $i = 1, 2$.

By Theorem 2.1, we consider, as usual, the inequalities that $|D : D'C_1| \leq M_1$ and $|C_1 : C'_1| \leq M_2$. If $|D : D'C_1| < M_1$ and $|C_1 : C'_1| < M_2$, we obtain a contradiction by (5).

Let us suppose that $|D : D'C_1| = M_1$. Since D/C_1 is nonabelian, we get $|(D/C_1)'| = 2$, that is, $D' \cap C_1 = 1$ because $|D'| = 2$. Also, since we have $C_1^g = C_2$ for $g \in G - D$

and $D'C_i$ is a normal subgroup of G for $i = 1, 2$, we obtain $D'C_1 = D'C_2$. This shows $C_1C_2 \leq D'C_1$. In fact, $C_1C_2 = D'C_1$ because $|D'C_1/C_1| = 2$ and C_1 and C_2 are disjoint subgroups of $D'C_1$. Since $|D'C_1/C_1| = 2$, we further obtain that $|C_1| = |C_2| = 2$, and thus $C_i \leq Z(D)$ for $i = 1, 2$. Since C_1 is abelian, it has a regular orbit on V_2 , which forces $M_2 = |C_1| = 2$ and, in particular, $|C_1 : C'_1| = M_2$. Let us pick $v_1 \in V_1$ such that $|v_1^D| = M_1$. Hence we have

$$|D : D'C_1| = |D/C_1 : (D/C_1)'| = |v_1^D| = |D : C_D(v_1)|,$$

which shows that $|C_D(v_1)| = |D'C_1| = 2|C_1| = 4$, and hence $C_D(v_1)$ is abelian. Assume that the action of $C_D(v_1)$ on V_2 is not faithful. Since $|C_2| = 2$, then $C_2 \leq C_D(v_1)$. As clearly $C_1 \leq C_D(v_1)$, altogether we have $C_D(v_1) = C_1C_2 \leq Z(D)$, which obviously is not possible. Thus $C_D(v_1)$ acts faithfully on V_2 . Then the abelian group $C_D(v_1)$ has a regular orbit $v_2^{C_D(v_1)}$ on V_2 for some $v_2 \in V_2$. Since

$$|D : C_D(v_1 + v_2)| = |D : C_D(v_1) \cap C_D(v_2)| = |D : C_D(v_1)| |v_2^{C_D(v_1)}| = |D|,$$

we deduce that $v_1 + v_2$ is a regular orbit of D on V_D . This is a contradiction because by Lemma 2.7(a) we have $M_D = M_1M_2 = |D : D'C_1||C_1| = (|D|/4) \cdot 2 = |D|/2$.

We now consider the last case where $|D : D'C_1| < M_1$ and $|C_1 : C'_1| = M_2$. By (6), we have $M_1 = 2|D/C_1 : (D/C_1)'| = |D/C_1|$, and hence D/C_1 has a regular orbit on V_1 . Moreover, since $D'C_1 = D'C_2 = C_1C_2$ and $|D'C_1/C_1| = 2$, we conclude that C_1 and C_2 are of order 2. Also, $C_1 \cong C_1C_2/C_2$ is a subgroup of $Z(D/C_2)$, which acts frobeniusly on V_2 , so C_1 acts frobeniusly on V_2 . Now let $g \in G - D$. Assume that D has at least two nontrivial orbits O_1 and O_2 on V_1 . We may assume that O_1 is a regular orbit of D/C_1 , that is, $|O_1| = |D|/2$. Note that then O_1^g and O_2^g are two distinct orbits of D on V_2 . Now let $x_1 \in O_1$ and $x_2 \in O_2^g$, so in particular $x_1 \in V_1$ and $x_2 \in V_2$. Since x_1 is in a regular orbit of D/C_1 on V_1 , we conclude that $C_D(x_1) = C_1$ and $C_{C_1}(x_2) = 1$, and so if we let $v = x_1 + x_2$, then v is in a regular orbit of D on V . Since G has no regular orbit on V and $|D| = |v^D| \leq |v^G| < |G|$, we deduce that $|C_G(v)| = 2$, and so $C_G(v) = \langle y \rangle$ for some $y \in G - D$. Since G is a disjoint union of D and Dg , we can write $y = dg$ for some $d \in D$. We then have

$$v = v^y \in O_1^y + (O_2^g)^y = O_1^{dg} + (O_2^g)^{dg} = O_1^g + O_2^{g^{dg}} = O_1^g + O_2 = O_2 + O_1^g, \quad (9)$$

where the penultimate equality in Equation (9) follows since $gdg = g^2g^{-1}dg = g^2d^g \in D$ as $g^2 \in D$ and $d^g \in D$. Now $O_2 \subseteq V_1$ and $O_1^g \subseteq V_2$, so from Equation (9) we conclude that $x_1 \in O_2$. But recall that also $x_1 \in O_1$, so altogether we have $x_1 \in O_1 \cap O_2 = \emptyset$, which is a contradiction. This shows that D has exactly one nontrivial orbit on V_1 , and this orbit is a regular orbit of D/C_1 on V_1 . In other words, V_1 is a D/C_1 -module such that D/C_1 permutes the nonzero elements of V transitively. Hence Theorem 6.8 of [15] tells us that D/C_1 is a subgroup of the semilinear group $\Gamma(V_1)$, and furthermore $2^m = |D/C_1| = |V_1| - 1 = q^n - 1$ for a prime q and positive integers n and m . From

Proposition 3.1 of [15], we have that $n = 1$ and q is a Fermat prime or $n = 2$ and $m = q = 3$. If $n = 1$, then $\Gamma(V_1)$ becomes an abelian group. Since $D/C_1 \leq \Gamma(V_1)$, then we have a contradiction that D/C_1 is an abelian group. If $n = 2$ and $m = q = 3$, then $D/C_1 \cong D_8$. This leads to the contradiction that D_8 cannot have a regular orbit on V_1 . This final contradiction proves that $|\Phi(G)| = 2$.

Step 4. G is one of the groups in the theorem.

By Steps 3 and 4 we have $G' = \Phi(G)$. Since $Z(G)$ is cyclic and $G/\Phi(G)$ is an elementary abelian group, then we conclude that $|Z(G)| \leq 4$. By Lemma 4.2 of [3], we obtain that $G = EZ(G)$, where E is an extraspecial 2-group. It now follows from Lemma 1.5(i) of [16] that G is one of the groups in the theorem, and so we are done. $\square$

The reader should note that the group in the case (a), (b) and (c) of Theorem A has exactly two, three and five different orbits of maximal size $M = |G : G'|$ on the irreducible faithful module V , respectively. Thus, we confirm Theorem 1.2 and Conjecture 4.1 of [9] and Theorem 2 and Conjecture 1 of [10].

4. Proofs of Theorem B, Theorem C, and Corollary D

Before giving the proof of the theorems, we need a lemma, which could also be obtained by using GAP.

Lemma 4.1. *Let G be a finite group and let V be a finite faithful irreducible G -module. Suppose that G and V are one of the groups and modules in Theorem A. If $H < G$, then H has a regular orbit on V .*

Proof. We clearly may assume that H is a maximal subgroup of G . Then $|G : H| = 2$. If H is abelian, it has a regular orbit, and we are done. So suppose H is nonabelian, and, in particular, $G \not\cong D_8$. Then $|H'| = 2$. Note that all nontrivial orbits of G on V have size $|G|/2$. Now assume that H has no regular orbit on V . If $0 \neq v \in V$, then, as $|G : H| = 2$, we see that $|G|/2 = |v^G| \leq 2 \cdot |v^H|$. Thus, all nontrivial orbits size of H on V must have size $|H|/2$. So if $G \cong D_8 \circ C_4$, then V is an irreducible faithful H -module where the largest orbit size of H on V is $|H : H'|$ and $|H| = 8$, which gives us a contradiction by Theorem A. If $G \cong D_8 \circ Q_8$, then H acts completely reducibly on V with 10 orbits of size $8 = |H : H'|$, which is the largest orbit size of H on V . Hence Theorem B applies to the action of H on V and easily leads to a contradiction. $\square$

Proof of Theorem B. Put $M = |G : G'|$. We work by induction on $|GV|$. Clearly we may assume that G is nonabelian. We write $V = X \oplus Y$ for an irreducible G -submodule X such that $G/C_G(X)$ is nonabelian, and for a G -invariant complement Y of X in V . Note that $G/C_G(Y)$ acts faithfully and completely reducibly on Y with $C_Y(G/C_G(Y)) = 0$ and also, $|G/C_G(Y) : (G/C_G(Y))'|$ is the largest orbit size in the action of $G/C_G(Y)$ on Y by Lemma 2.3. We now apply induction to the action of $G/C_G(Y)$ on Y and

obtain that $G/C_G(Y)$ has the desired structure. In particular, $G/C_G(Y)$ is the direct product of n nonabelian groups G_i for $i = 1, \dots, n$, each of which is isomorphic to one of $D_8, D_8 \circ C_4, D_8 \circ Q_8$, and an abelian group A . Moreover, Y is the direct sum of n irreducible G -modules Y_i on which G_j acts nontrivially iff $i = j$, and some faithful A -module Y_{n+1} . In particular, if for any $i \in \{1, \dots, n+1\}$ we denote W_i to be the direct sum of all Y_j such that $j \neq i$, then we see that $G/C_G(Y)$ is the direct product of all $C_{G/C_G(Y)}(W_i)$ for $i = 1, \dots, n+1$.

First assume that $G' \cap C_G(Y) = 1$ and write $\bar{G} = G/C_G(Y)$. We note that then $C_G(Y)$ is contained in $Z(G)$ and as it is easily seen that $C_G(Y) \cong C_G(Y)C_G(X)/C_G(X)$ is a subgroup of $Z(G/C_G(X))$, and hence $C_G(Y)$ becomes a cyclic group. Since $C_G(Y)$ acts faithfully on X , we see that the cyclic group $C_G(Y)$ actually acts frobeniusly on X . For $i = 1, \dots, n$ we know that $C_{\bar{G}}(W_i)$ is isomorphic to one of $D_8, D_8 \circ C_4, D_8 \circ Q_8$. Let $y \in Y$ be in an orbit of maximal size of G on Y , that is, of size $|G : G' C_G(Y)| = |G : G'|/|C_G(Y)|$. Then $C_{\bar{G}}(y)$ is elementary abelian of order 2^n . Note that by Lemma 2.3 we have that $|\bar{G} : \bar{G}'| = |\bar{G} : C_{\bar{G}}(y)|$ and thus,

$$|G'| = |\bar{G}'| = |C_{\bar{G}}(y)| = 2^n.$$

More precisely, $|C_{C_{\bar{G}}(W_i)}(y)| = 2$ for $i = 1, \dots, n$, and clearly $C_{C_{\bar{G}}(W_i)}(y)$ is not normal in $C_{\bar{G}}(W_i)$.

Now put $C = C_G(y)$ and observe that since $C_{\bar{G}}(y)$ is abelian, we know that $C' \leq C_G(Y)$, and since $G' \cap C_G(Y) = 1$, it follows that C is abelian. Now consider the action of C on X . Since C is abelian, $C/C_C(X)$ has a regular orbit on X . Hence if $x \in X$ is in such an orbit and if we let $v = x + y$, then

$$C_G(v) = C_G(x) \cap C = C_C(x) = C_C(X) \cong C_C(X)C_G(Y)/C_G(Y)$$

is elementary abelian of order at most 2^n , and $C_G(v) \leq C_G(X)$. If the order is strictly smaller than 2^n , then $|C_G(v)| < |G'|$, which contradicts that $|G : G'|$ is the largest orbit size of G on V . Thus $|C_G(v)| = 2^n$ and $C_G(v)$ is elementary abelian. Since $C_G(v) \leq C_G(X)$, clearly $C_G(v) \cap C_G(Y) = 1$, and hence $C_G(v) \cong C_G(v)C_G(Y)/C_G(Y) \leq C_{\bar{G}}(y)$. On the other hand, since all groups in the last inequality have the same order 2^n , we obtain that $C_G(v) \cong C_{\bar{G}}(y)$ and the actions of these groups on Y are equivalent.

So we know that $C_G(v) \leq C_G(X) \trianglelefteq G$. Therefore $C_G(X) \cap C_G(W_i)$ is a normal subgroup of $C_G(W_i)$ for $i = 1, \dots, n$ which contains $C_{C_G(W_i)}(v) \cong C_{C_{\bar{G}}(W_i)}(y)$, which is of order 2. In particular, $C_G(X) \cap C_G(W_i) > 1$, and hence $C_G(X) \cap C_G(W_i)$ contains $G' \cap C_G(W_i)$, which is of order 2. Since this is true for all $i = 1, \dots, n$, it follows that $G' \leq C_G(X)$. However, we know that $G/C_G(X)$ is nonabelian and hence $1 < (G/C_G(X))' = G' C_G(X)/C_G(X)$, which implies that G' is not a subgroup of $C_G(X)$. This contradiction shows that $G' \cap C_G(Y) = 1$ cannot happen. So from now on we know that $G' \cap C_G(Y) > 1$.

Now let $y \in Y$ be in an orbit of maximal size of G on Y . Then $C_{G/C_G(Y)}(y)$ is elementary abelian of order 2^n , and since by Lemma 2.3 we know that

$$|C_{G/C_G(Y)}(y)| = |(G/C_G(Y))'| = |G'C_G(Y)/C_G(Y)| = |G'/(G' \cap C_G(Y))| < |G'|. \quad (10)$$

Now assume that $C_G(Y) \cong C_G(Y)C_G(X)/C_G(X) < G/C_G(X)$. By Lemma 4.1, we know that there is an $x \in X$ such that $C_{C_G(Y)}(x) = 1$. Thus $C_G(x)$ is isomorphic to a subgroup of $G/C_G(Y)$. Hence if we let $v = x + y$, then $C_G(v) = C_G(x) \cap C_G(y) = C_{C_G(x)}(y)$ is isomorphic to a subgroup of $G/C_G(Y)$, and hence $|C_G(v)| \leq |C_{G/C_G(Y)}(y)| < |G'|$, where the last inequality follows from (10). This shows that $M \geq |v^G| = |G : C_G(v)| > |G : G'|$, a contradiction. Therefore $C_G(Y) \cong C_G(Y)C_G(X)/C_G(X) = G/C_G(X)$, and hence $|G| = |C_G(Y)||C_G(X)|$ and clearly $C_G(Y) \cap C_G(X) = 1$. Thus $G = C_G(Y)C_G(X) = C_G(Y) \times C_G(X)$ and $C_G(X) \cong G/C_G(Y)$, and the action of $C_G(Y) \cong G/C_G(X)$ on X has the desired structure by Theorem A. Altogether the theorem follows. $\square$

Remark 4.2. Theorem B provides another proof of Theorems 1.4 and 1.5 of [4].

We now give the proof of Theorem C and Corollary D.

Proof of Theorem C. By Theorem 2.8 of [18], we know that $|G : G'| \leq M$, where M is the largest orbit size in the action of G on V and Theorem 2.1 remains true for all finite groups when G acts faithfully on arbitrary finite vector space V with $\text{char}(V) = p$ and $O_p(G) = 1$ for a prime p . Since $|G : G'| = M$, we thus have that G is a nilpotent group. Since $O_p(G) = 1$ and G is nilpotent, we conclude that p does not divide the order of G . It follows that V becomes a completely reducible G -module. Considering the fact that $\text{char}(V) = p$ and Theorem B, we get the desired result. $\square$

Proof of Corollary D. We know from Corollary 13 of [21] that if $I(B)'$ has prime order or order 4, then $k(B) \leq |D|$. Thus, we may assume that $|I(B)'| = 6$. By Theorem 2.1, if we choose an element $x \in D$ in a maximal orbit of $I(B)$ on D , then $|I(B) : I(B)'| \leq |I(B) : C_{I(B)}(x)|$, which gives us that $|C_{I(B)}(x)| \leq |I(B)'| = 6$. If $|C_{I(B)}(x)| < |I(B)'| = 6$, then by Lemma 14.5 of [20] and Proposition 11 of [21] we have $k(B) \leq |D|$. Therefore, we may assume that $|C_{I(B)}(x)| = |I(B)'| = 6$, which implies that $|I(B) : I(B)'|$ is the maximal orbit size in the action of $I(B)$ on D . By Theorem 2.1, $I(B)$ is abelian or is a nilpotent group having at least two different orbits of maximal size on D . Since $I(B)$ is nonabelian, we have the latter case. By Theorem B, we have that $I(B)'$ is an elementary abelian 2-group, contradicting $|I(B)'| = 6$. $\square$

5. Outlook

We conclude this paper by making a few remarks on possible future research directions related to our results in this paper. First note that the groups we obtain in our results are all 2-groups of class two satisfying some special conditions. We arrived at them by imposing an extremal orbit condition on a linear group action. One now could go the other way round and ask: If we impose some special conditions on the groups, what does

this mean for the orbit sizes in their linear actions on finite vector spaces? For example, in [5] the authors study p -groups which satisfy what they call the *strong condition on normal subgroups*. Without going into detail here, we mention that they obtain restrictions in the character degrees of such groups, that is, the dimensions of the modules over $\mathbb{C}$ they act on linearly. The groups in our Theorem A satisfy the strong condition. Therefore it might be interesting to study whether the strong condition has any influence on the orbit structure of the groups on finite modules.

In a similar vein, we already observed in the Introduction that Theorem A can be viewed as a generalization of a result by Passman [16, Lemma 1.5(i)] on the orbits of special groups called *groups of type $E(p, m)$* . It characterizes when such groups have no regular orbits in a linear group action. Now Part (ii) of [16, Lemma 1.5] characterizes those linear group actions of groups G of type $E(p, m)$ which have an orbit of size $|G|/p$, and again this forces $p = 2$, and there are only three very special actions satisfying this condition. Is there a generalization of this result to arbitrary finite groups similar in spirit to the generalization that Theorem A is to Part(i) of [16, Lemma 1.5]?

Data availability

No data was used for the research described in the article.

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