Open Mathematics

Research Article

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On a conjecture about generalized Q-recurrence

https://doi.org/10.1515/math-2018-0124
Received June 19, 2018; accepted November 23, 2018.

Abstract: Chaotic meta-Fibonacci sequences which are generated by intriguing examples of nonlinear recurrences still keep their mystery although substantial progress has been made in terms of well-behaved solutions of nested recurrences. In this study, a recent generalization of Hofstadter's famous Q-sequence is studied beyond the known methods of generational approaches in order to propose a generalized conjecture regarding the existence of infinitely many different solutions for all corresponding recurrences of this generalization.

Keywords: Hofstadter Q-recurrence, Meta-Fibonacci Sequences, Nonlinear recurrences

MSC: 11B37

1 Introduction

Since substantial experimental studies have been done by Hofstadter and Huber on certain meta-Fibonacci sequences which are inspired by original Q-sequence that is defined by $Q(n) = Q(n-Q(n-1)) + Q(n-Q(n-2))$ with initial conditions $Q(1) = Q(2) = 1$ [1], there is a wide variety of significant contributions which are mainly focused on nested recurrence relations in theoretical and empirical literature [2-10]. While empirical works try to find order signs and some patterns in chaotic meta-Fibonacci sequences, theoretical literature mainly focuses on properties of slow and quasi-periodic solutions for various types of nested recursions and some curious concepts such as binary trees and undecidability [11–13]. Despite the complexity of many inquiries which these contributions have, an essential and natural question about a meta-fibonacci sequence is whether it is defined for all indices [14, 15]. It is well-known that properties of solutions to meta-Fibonacci recursions can differ extremely and the answer of this question strongly depends on the selection of initial conditions in general [16]. While modified nested recurrences can exhibit different properties of solutions with the same initial conditions, a selected recurrence can also show dissimilar behaviours under certain initial condition sets [17, 18]. For example, let us define $Q(1) = Q(2) = Q(3) = 1$, $Q(4) = 4$, $Q(5) = 3$, and for $n \geq 5$, $Q(n) = Q(n-Q(n-1)) + Q(n-Q(n-2)) + Q(n-Q(n-3))$, that is A296413 in OEIS [19]. This sequence is highly chaotic and finite since $Q(509871) = 519293$. On the other hand, the same recurrence can have slow solution with appropriate initial conditions [20] while it can be naturally quasi-periodic for certain set of initial conditions such as A296518 and A296786 in OEIS [21]. Also, intriguing generational structures which are analysed by spot-based generation concept appear with a family of initial conditions for $Q(n) = Q(n-Q(n-1)) + Q(n-Q(n-2)) + Q(n-Q(n-3))$ [21]. This behavioral diversity observed in three-term Hofstadter Q-recurrence can be interpreted as a sign of difficulty of problems about certain types of nested recurrences, especially for Hofstadter-like recurrences. On the other hand, the coexistence of order

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and chaos in some unpredictable meta-Fibonacci sequences can be recreated experimentally thanks to inital condition patterns in terms of considerable behavioral similarities. At this point, it would be beneficial to remember the definition of a generalization of Hofstadter's Q-sequence [21].

Definition 1.1. *Let $Q_{d,\ell}(n)$ be defined by the recurrence $Q_{d,\ell}(n) = \sum_{i=1}^{\ell} Q_{d,\ell}(n - Q_{d,\ell}(n-i))$ for $n > d \cdot \ell$, $\ell \geq 2$, $d \geq 1$, with the initial conditions $Q_{d,\ell}(n) = \lceil \frac{n \cdot (\ell-1)}{\ell} \rceil$ for $1 \leq n \leq d \cdot \ell$.*

Recently, the chaotic generational structures of certain members of $Q_{d,2}(n)$ and $Q_{d,3}(n)$ were investigated thanks to known methods in empirical literature [21–23] in order to search a conjectural global property for rescaling of amplitudes of successive generations. On the other hand, for ascending values of d and ℓ, behaviours of $Q_{d,\ell}(n)$ become much more complicated as well as conserving their interesting properties which are reported in the following sections. At the same time, in order to strengthen and support our intuitive approach on nature of initial conditions for $Q_{d,\ell}(n)$, it can be beneficial to observe behaviours of similar initial condition formulation with a provable example.

This paper is structured as follows. In Section 1.1, we give brief notes about an intriguing example of provable solution family in order to support the naturalness of our intuition. In Section 2, we prove some properties of $Q_{d,\ell}(n)$ for increasing values of ℓ. Then, in Section 3, we do experiments for larger values of d and ℓ while in Section 4, we propose a conjecture thanks to results that Section 2 and Section 3 provide. Finally, related concluding remarks are offered in Section 5.

1.1 A motivational new generalization to beginning

In this section, we observe the potential of initial condition patterns in order to construct infinitely many different solutions for a selected nested recurrence. Although sequences that we focus on in next sections are not slow, this part suggests that infinitely many different solutions that have strong behavioral similarities can be really constructed with the formulation of initial conditions, at least for certain recurrences. It is well-known that Newman generalization on Hofstadter-Conway \$10000 sequence (A004001 in OEIS) provides a variety of interesting results that Fibonacci-type behaviours appear in their generational analysis [23–25]. However, we will generalize Hofstadter-Conway \$10000 sequence thanks to its asymptotic property which is proved by Conway [24]. In order to eliminate duplicate sequences, we define them as below. Note that $a_1(n)$ is Conway's original sequence.

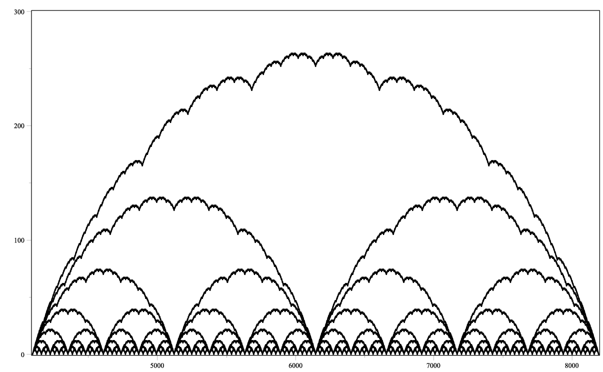
Definition 1.2. *Let $a_i(n) = a_i(a_i(n-1)) + a_i(n - a_i(n-1))$ for $n > 4 \cdot i$, with the initial conditions $a_i(n) = \lceil \frac{n}{2} \rceil$ for $1 \leq n \leq 4 \cdot i$.*

Proposition 1.3. *$a_i(n+1) - a_i(n) \in \{0, 1\}$ for all $n, i \geq 1$ and $a_i(n)$ hits every positive integer for all $i \geq 1$.*

Proposition 1.4. *$a_i(n) \geq \frac{n}{2}$ for all $i \geq 1$.*

Both propositions are easy to prove by induction since initial conditions of the form $\lceil \frac{n}{2} \rceil$ provide necessary and sufficient basis to this aim. Based on these basic properties, see Figure 1 in order to observe certain members of $a_i(n) - \frac{n}{2}$. While it provides a fractal-like collection of fractal-like sequences, it clearly suggests that this is a new and natural generalization which perfectly guarantees the existence of infinitely many different solutions and our approach can be potentially applied for some other recurrences. In next sections we search for the signs of some similar behaviours in terms of existence of solutions for $Q_{d,\ell}(n)$ with the stability of noise levels, although their chaotic nature prevents the perfection which $a_i(n)$ exhibits.

Fig. 1. $a_{2^t}(n) - \frac{n}{2}$ for $2^{12} \le n \le 2^{13}$ and $0 \le t \le 7$, respectively.



2 On termination of $Q_{d,\ell}(n)$ for $\ell \ge 3$

In this section, an existence of breaking point for termination of $Q_{d,\ell}(n)$ is investigated. It is a relatively easy fact that $Q_{2,3}(n)$ is a finite sequence due to $Q_{2,3}(66) = 73$ [21]. On the other hand, our following analysis confirms that the termination of $Q_{2,3}(n)$ is an exceptional behaviour for $Q_{d,\ell}(n)$, at least, in the range of experiments that the next section focuses on.

Proposition 2.1. *$Q_{d,\ell}(n)$ dies for $d < \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$ where $\ell \ge 3$.*

Proof. It is clear that if $Q_{d,\ell}(d\cdot\ell+1) > d\cdot\ell+1$, then $Q_{d,\ell}(n)$ dies immediately thereafter. Since $Q_{d,\ell}(n) = \left\lceil \frac{n\cdot(\ell-1)}{\ell} \right\rceil$ for $n \le d\cdot\ell$,

$$\begin{aligned} Q_{d,\ell}(d\cdot\ell+1-i) &= \left\lceil \frac{(d\cdot\ell+1-i)\cdot(\ell-1)}{\ell} \right\rceil \\ &= \left\lceil d\cdot\ell+1-(d+i)+\frac{i-1}{\ell} \right\rceil \\ &= d\cdot\ell+1-(d+i)+\left\lceil \frac{i-1}{\ell} \right\rceil \end{aligned}$$

for $1 \le i \le d\cdot\ell$.

So $Q_{d,\ell}(d\cdot\ell+1)$ can be expressed as follows:

$$\begin{aligned} Q_{d,\ell}(d\cdot\ell+1) &= \sum_{i=1}^{\ell} Q_{d,\ell}(d\cdot\ell+1-Q_{d,\ell}(d\cdot\ell+1-i)) \\ &= \sum_{i=1}^{\ell} Q_{d,\ell}\left(d\cdot\ell+1-\left(d\cdot\ell+1-(d+i)+\left\lceil \frac{i-1}{\ell} \right\rceil\right)\right) \\ &= \sum_{i=1}^{\ell} Q_{d,\ell}\left(d+i-\left\lceil \frac{i-1}{\ell} \right\rceil\right) \\ &= Q_{d,\ell}(d+1)+\sum_{i=1}^{\ell-1} Q_{d,\ell}(d+i) \\ &= \left\lceil \frac{(d+1)\cdot(\ell-1)}{\ell} \right\rceil+\sum_{i=1}^{\ell-1}\left\lceil \frac{(d+i)\cdot(\ell-1)}{\ell} \right\rceil \\ &= d+1-\left\lfloor \frac{d+1}{\ell} \right\rfloor+\sum_{i=1}^{\ell-1}(d+i)-\sum_{i=1}^{\ell-1}\left\lfloor \frac{d+i}{\ell} \right\rfloor \\ &= d\cdot\ell+1+\frac{\ell\cdot(\ell-1)}{2}-\left\lfloor \frac{d+1}{\ell} \right\rfloor-\sum_{i=1}^{\ell-1}\left\lfloor \frac{d+i}{\ell} \right\rfloor. \end{aligned}$$

So, the inequality $Q_{d,\ell}(d \cdot \ell + 1) > d \cdot \ell + 1$ is equivalent to the inequality

$$\left\lfloor \frac{d+1}{\ell} \right\rfloor + \sum_{i=1}^{\ell-1} \left\lfloor \frac{d+i}{\ell} \right\rfloor < \frac{\ell \cdot (\ell - 1)}{2}. \tag{1}$$

If $d = \frac{\ell \cdot (\ell-1)}{2}$ in Inequality 1, total value of ℓ terms of the left side is $\frac{\ell \cdot (\ell-1)}{2}$ for $l \geq 3$, and the inequality is not satisfied. Note that $\frac{\ell \cdot (\ell-1)}{2} \equiv 0 \pmod{\ell}$ if ℓ is odd and $\frac{\ell \cdot (\ell-1)}{2} \equiv \frac{\ell}{2} \pmod{\ell}$ if ℓ is even. So, if $d = \frac{\ell \cdot (\ell-1)}{2} - 1$, the left side decreases from $\frac{\ell \cdot (\ell-1)}{2}$ if ℓ is even (thereby satisfying the inequality), but the left side does not change from before if ℓ is odd. In the odd case, if $d = \frac{\ell \cdot (\ell-1)}{2} - 2$, the left side would decrease, though. So, if ℓ is odd, the largest value of d that satisfies Inequality 1 is $\frac{\ell \cdot (\ell-1)}{2} - 2$, and if ℓ is even, the largest value of d that satisfies Inequality 1 is $\frac{\ell \cdot (\ell-1)}{2} - 1$. These two cases can be combined into one inequality: $d < \frac{\ell \cdot (\ell-1) - 1 + (-1)^{\ell}}{2}$. □

Proposition 2.2. $Q_{d,\ell}(d \cdot \ell + 2) = d \cdot (\ell - 1) + \lfloor \frac{(\ell+3)}{2} \rfloor$ *for* $d = \frac{\ell \cdot (\ell-1) - 1 + (-1)^{\ell}}{2}$ *where* $\ell \geq 5$.

Proof. We follow similar steps with previous proof thanks to its result. So $Q_{d,\ell}(d \cdot \ell + 2)$ can be expressed as follows:

$$\begin{aligned}
Q_{d,\ell}(d \cdot \ell + 2) &= \sum_{i=1}^{\ell} Q_{d,\ell}(d \cdot \ell + 2 - Q_{d,\ell}(d \cdot \ell + 2 - i)) \\
&= Q_{d,\ell}(d \cdot \ell + 2 - Q_{d,\ell}(d \cdot \ell + 1)) + \sum_{i=2}^{\ell} Q_{d,\ell}(d \cdot \ell + 2 - Q_{d,\ell}(d \cdot \ell + 2 - i)) \\
&= Q_{d,\ell}(1) + \sum_{i=2}^{\ell} Q_{d,\ell}\left(d \cdot \ell + 2 - \left(d \cdot \ell + 2 - (d+i) + \left\lceil \frac{i-2}{\ell} \right\rceil\right)\right) \\
&= 1 + \sum_{i=2}^{\ell} Q_{d,\ell}\left(d + i - \left\lceil \frac{i-2}{\ell} \right\rceil\right) \\
&= 1 + Q_{d,\ell}(d+2) + \sum_{i=2}^{\ell-1} Q_{d,\ell}(d+i) \\
&= 1 + \left\lceil \frac{(d+2) \cdot (\ell-1)}{\ell} \right\rceil + \sum_{i=2}^{\ell-1} \left\lceil \frac{(d+i) \cdot (\ell-1)}{\ell} \right\rceil \\
&= d + 3 - \left\lfloor \frac{d+2}{\ell} \right\rfloor + \sum_{i=2}^{\ell-1} (d+i) - \sum_{i=2}^{\ell-1} \left\lfloor \frac{d+i}{\ell} \right\rfloor \\
&= d \cdot (\ell - 1) + 2 + \frac{\ell \cdot (\ell-1)}{2} - \left\lfloor \frac{d+2}{\ell} \right\rfloor - \sum_{i=2}^{\ell-1} \left\lfloor \frac{d+i}{\ell} \right\rfloor.
\end{aligned}$$

Since $\lfloor \frac{d+1}{\ell} \rfloor = \lfloor \frac{d+2}{\ell} \rfloor$ for $l \geq 5$ and $d = \frac{\ell \cdot (\ell-1) - 1 + (-1)^{\ell}}{2}$, from Proposition 2.1,

$$\frac{\ell \cdot (\ell-1)}{2} - \left\lfloor \frac{d+2}{\ell} \right\rfloor - \sum_{i=2}^{\ell-1} \left\lfloor \frac{d+i}{\ell} \right\rfloor = \left\lfloor \frac{d+1}{\ell} \right\rfloor. \tag{2}$$

Thanks to Equation 2, $Q_{d,\ell}(d \cdot \ell + 2)$ can be expressed as follows for $l \geq 5$,

$$\begin{aligned}
Q_{d,\ell}(d \cdot \ell + 2) &= d \cdot (\ell - 1) + 2 + \left\lfloor \frac{d+1}{\ell} \right\rfloor \\
&= d \cdot (\ell - 1) + 2 + \left\lfloor \frac{\frac{\ell \cdot (\ell-1) - 1 + (-1)^{\ell}}{2} + 1}{\ell} \right\rfloor \\
&= d \cdot (\ell - 1) + \lfloor \frac{(\ell+3)}{2} \rfloor.
\end{aligned}$$

□

Proposition 2.3. *$Q_{d,\ell}(d\cdot\ell+3)=d\cdot(\ell-1)+\frac{7+(-1)^{\ell+1}}{2}$ for $d=\frac{\ell\cdot(\ell-1)-1+(-1)^{\ell}}{2}$ where $\ell\geq 7$.*

Proof. We follow similar steps with the previous proof thanks to result of it. So $Q_{d,\ell}(d\cdot\ell+3)$ can be expressed as follows:

$$\begin{aligned}
Q_{d,\ell}(d\cdot\ell+3)&=\sum_{i=1}^{\ell}Q_{d,\ell}(d\cdot\ell+3-Q_{d,\ell}(d\cdot\ell+3-i))\\
&=Q_{d,\ell}(d\cdot\ell+3-Q_{d,\ell}(d\cdot\ell+2))+Q_{d,\ell}(d\cdot\ell+3-Q_{d,\ell}(d\cdot\ell+1))\\
&\quad+\sum_{i=3}^{\ell}Q_{d,\ell}(d\cdot\ell+3-Q_{d,\ell}(d\cdot\ell+3-i))\\
&=Q_{d,\ell}(d\cdot\ell+3-d\cdot(l-1)-\lfloor\frac{(l+3)}{2}\rfloor)+Q_{d,\ell}(2)\\
&\quad+\sum_{i=3}^{\ell}Q_{d,\ell}\Big(d\cdot\ell+3-\Big(d\cdot\ell+3-(d+i)+\Big\lceil\frac{i-3}{\ell}\Big\rceil\Big)\Big)\\
&=2+Q_{d,\ell}(d+3-\lfloor\frac{(\ell+3)}{2}\rfloor)+\sum_{i=3}^{\ell}Q_{d,\ell}\Big(d+i-\Big\lceil\frac{i-3}{\ell}\Big\rceil\Big)\\
&=2+Q_{d,\ell}(d+3-\lfloor\frac{(\ell+3)}{2}\rfloor)+Q_{d,\ell}(d+3)+\sum_{i=3}^{\ell-1}Q_{d,\ell}(d+i)\\
&=2+Q_{d,\ell}(d+3-\lfloor\frac{(\ell+3)}{2}\rfloor)+\Big\lceil\frac{(d+3)\cdot(\ell-1)}{\ell}\Big\rceil\\
&\quad+\sum_{i=3}^{\ell-1}\Big\lceil\frac{(d+i)\cdot(\ell-1)}{\ell}\Big\rceil\\
&=d+5+Q_{d,\ell}(d+3-\lfloor\frac{(l+3)}{2}\rfloor)-\Big\lfloor\frac{d+3}{\ell}\Big\rfloor+\sum_{i=3}^{\ell-1}(d+i)-\sum_{i=3}^{\ell-1}\Big\lfloor\frac{d+i}{\ell}\Big\rfloor\\
&=d\cdot(\ell-2)+2+Q_{d,\ell}(d+3-\lfloor\frac{(l+3)}{2}\rfloor)+\frac{\ell\cdot(\ell-1)}{2}-\Big\lfloor\frac{d+3}{\ell}\Big\rfloor\\
&\quad-\sum_{i=3}^{\ell-1}\Big\lfloor\frac{d+i}{\ell}\Big\rfloor.
\end{aligned}$$

Since $\lfloor\frac{d+2}{\ell}\rfloor=\lfloor\frac{d+3}{\ell}\rfloor$ for $l\geq 7$ and $d=\frac{\ell\cdot(\ell-1)-1+(-1)^{\ell}}{2}$, from Proposition 2.2,

$$\frac{\ell\cdot(\ell-1)}{2}-\Big\lfloor\frac{d+3}{\ell}\Big\rfloor-\sum_{i=3}^{\ell-1}\Big\lfloor\frac{d+i}{\ell}\Big\rfloor=\Big\lfloor\frac{d+1}{\ell}\Big\rfloor+\Big\lfloor\frac{d+2}{\ell}\Big\rfloor. \tag{3}$$

Thanks to Equation 3, $Q_{d,\ell}(d\cdot\ell+3)$ can be expressed as follows for $l\geq 7$,

$$\begin{aligned}
Q_{d,\ell}(d\cdot\ell+3)&=d\cdot(\ell-2)+Q_{d,\ell}(d+3-\lfloor\frac{(l+3)}{2}\rfloor)+2+\Big\lfloor\frac{d+1}{\ell}\Big\rfloor+\Big\lfloor\frac{d+2}{\ell}\Big\rfloor\\
&=d\cdot(\ell-2)+Q_{d,\ell}(d+3-\lfloor\frac{(l+3)}{2}\rfloor)+2+\Bigg\lfloor\frac{\frac{\ell\cdot(\ell-1)-1+(-1)^{\ell}}{2}+1}{\ell}\Bigg\rfloor\\
&\quad+\Bigg\lfloor\frac{\frac{\ell\cdot(\ell-1)-1+(-1)^{\ell}}{2}+2}{\ell}\Bigg\rfloor\\
&=d\cdot(\ell-2)+Q_{d,\ell}(d+3-\lfloor\frac{(\ell+3)}{2}\rfloor)+2\cdot\lfloor\frac{(\ell+1)}{2}\rfloor.
\end{aligned}$$

In here if ℓ is odd, then $d=\frac{\ell\cdot(\ell-1)}{2}-1$ and $Q_{d,\ell}(d\cdot\ell+3)$ can be expressed as follows,

$$\begin{aligned}
Q_{d,\ell}(d\cdot\ell+3)&=d\cdot(\ell-2)+Q_{d,\ell}(\frac{(\ell-1)^2}{2})+\ell+1\\
&=d\cdot(\ell-2)+\Big\lceil\frac{(\ell-1)^3}{2\cdot\ell}\Big\rceil+\ell+1
\end{aligned}$$

$$\begin{aligned}&= d \cdot (\ell - 2) + \frac{\ell \cdot (\ell - 3)}{2} + \ell + 3 \\ &= d \cdot (\ell - 2) + \frac{\ell \cdot (\ell - 1)}{2} + 3 \\ &= d \cdot (\ell - 1) + 4.\end{aligned}$$

And if ℓ is even, then $d = \frac{\ell \cdot (\ell - 1)}{2}$ and similar way can be used,

$$\begin{aligned}Q_{d,\ell}(d \cdot \ell + 3) &= d \cdot (\ell - 2) + Q_{d,\ell}(\frac{(\ell - 1)^2 + 3}{2}) + \ell \\ &= d \cdot (\ell - 2) + \frac{\ell \cdot (\ell - 3)}{2} + \ell + 3 \\ &= d \cdot (\ell - 2) + \frac{\ell \cdot (\ell - 1)}{2} + 3 \\ &= d \cdot (\ell - 1) + 3.\end{aligned}$$

These two cases can be combined into one equation that is $Q_{d,l}(d \cdot \ell + 3) = d \cdot (\ell - 1) + \frac{7 + (-1)^{\ell+1}}{2}$. □

Since each computed term $Q_{d,\ell}(d \cdot \ell + k)$ is also determinative for $Q_{d,\ell}(d \cdot \ell + k + 1)$ for all $k \geq 1$ by definition of our nested recurrences, the properties of further terms can also be shown with similar propositions. Table 1 contains some related patterns which guarantee the successive computations of recursions without termination in these short intervals as below. See also decreasing order of corresponding bursts in Figure 2 for an example $Q_{209,21}(n)$.

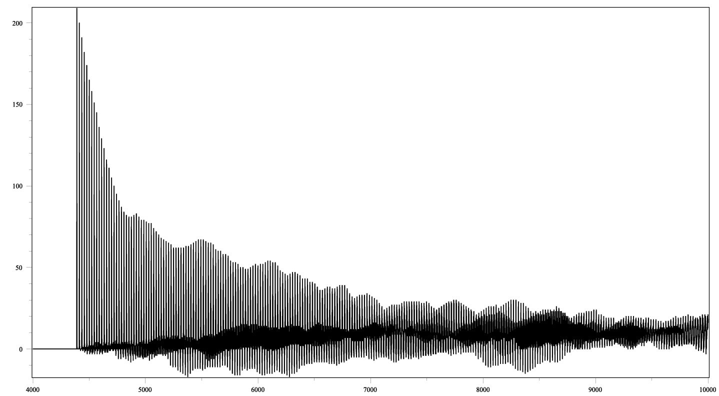
Table 1. Behaviour of $Q_{d,\ell}(d \cdot \ell + k)$ for $\ell \geq 2 \cdot k + 1$ and $k \leq 10$ where $d = \frac{\ell \cdot (\ell - 1) - 1 + (-1)^{\ell}}{2}$.

k	$Q_{d,\ell}(d \cdot \ell + k) - Q_{d,\ell}(d \cdot \ell)$	$Q_{d,\ell}(d \cdot \ell + k)$
1	$(\ell \cdot (\ell - 1) + 1 + (-1)^{\ell})/2$	$(\ell^3 - \ell^2 - \ell + \ell \cdot (-1)^{\ell} + 2)/2$
2	$\lfloor (\ell + 3)/2 \rfloor$	$(2 \cdot \ell^3 - 4 \cdot \ell^2 + 2 \cdot \ell + (2 \cdot \ell - 3) \cdot (-1)^{\ell} + 7)/4$
3	$(7 + (-1)^{\ell+1})/2$	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 2) \cdot (-1)^{\ell} + 8)/2$
4	$(9 + (-1)^{\ell+1})/2$	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 2) \cdot (-1)^{\ell} + 10)/2$
5	5	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 1) \cdot (-1)^{\ell} + 11)/2$
6	$(11 + (-1)^{\ell})/2$	$(\ell^3 - 2 \cdot \ell^2 + \ell \cdot (-1)^{\ell} + 12)/2$
7	7	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 1) \cdot (-1)^{\ell} + 15)/2$
8	8	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 1) \cdot (-1)^{\ell} + 17)/2$
9	9	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 1) \cdot (-1)^{\ell} + 19)/2$
10	10	$(\ell^3 - 2 \cdot \ell^2 + (\ell - 1) \cdot (-1)^{\ell} + 21)/2$

3 Experiments for Strange Members of $Q_{d,\ell}(n)$

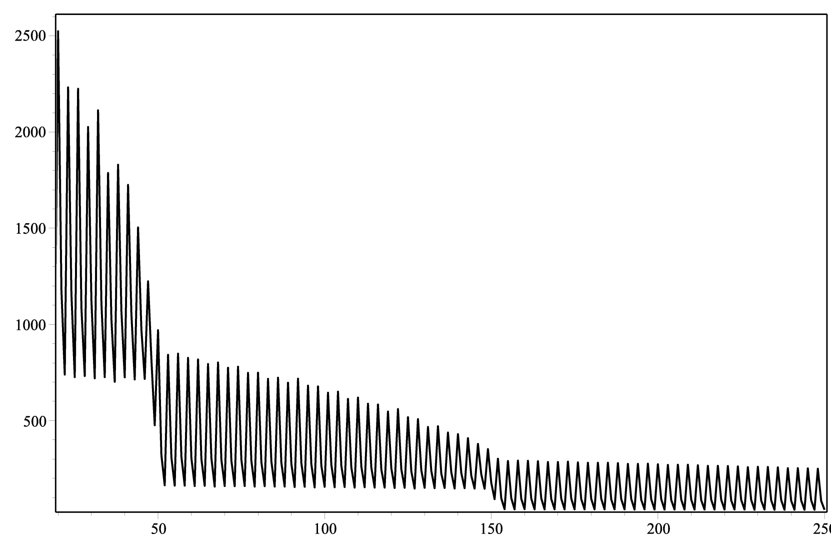
Although previous section is really helpful in order to understand the certain facts about possible termination cases of $Q_{d,\ell}(n)$ sequences for increasing values of ℓ and its relation with d, it is practically impossible to continue to prove these similar successive propositions forever. At this point, power of computer experiments may be meaningful in order to see the big picture about behaviours of these chaotic sequences in relatively long run and indeed this is the case. Our first purpose is the investigation of chaotic behaviours of these sequences for increasing values of $d \geq \frac{\ell \cdot (\ell - 1) - 1 + (-1)^{\ell}}{2}$ for a selected ℓ. To this aim, let $\langle S_{d,\ell}(n) \rangle_N$ denotes the average value of $S_{d,\ell}(n)$ for $d \cdot \ell + 1 \leq n \leq N + d \cdot \ell$ where $S_{d,\ell}(n)$ is defined by Equation 4 and $\sigma_{\ell}(d)$ is defined by Equation 5. In our experiments, we focus on $N = 10^6$, $2 \leq \ell \leq 19$ although several other values are also checked. Figure 3 particularly shows fascinating order signs of $\sigma_3(d)$ for such a chaotic

Fig. 2. Graph of $Q_{209,21}(n) - \frac{20}{21} \cdot n$ for $4000 \leq n \leq 10000$.



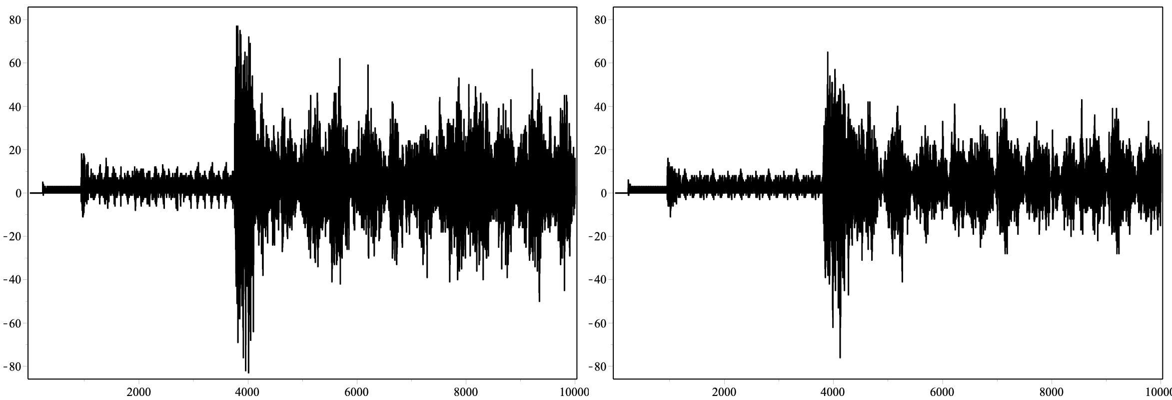
sequence family $Q_{d,3}(n)$. While $\sigma_3(d)$ has a very characteristic distributional behaviour based on d mod 3, meaningful observations also can be made in Figure 5 and Figure 6 that suggest behaviour of these sequences has considerable similarities in terms of fluctuations of $\sigma_\ell(d)$.

Fig. 3. Graph of $\sigma_3(d)$ for $20 \leq d \leq 250$.



This is a strong evidence that there are different branches in scatterplots which are dominated by values of d in a particular residue class for such chaotic sequences. Indeed, other graphs have also some similar patterns which are more irregular and weaker than patterns of $\sigma_3(d)$ in terms of their ℓ values. While perfect explanation of such distributions is very hard to formulate due to chaotic nature of them, the main common property of them is that they do not tend to increase in terms of general trends. More precisely, trend line slopes for selected intervals are negative for all of them in the range of our experiments. Additionally, there are some sharp transitions between some levels of $\sigma_\ell(d)$. In order to observe the meaning of a characteristic decrease example in $\sigma_4(d)$ plot, see Figure 4 that displays line graphs of $S_{59,4}(n)$ and $S_{60,4}(n)$ which small increase of d notably diminishes the amplitude of generational oscillations. Such a graphical observation is named as *Yosemite Graph* similarly in another curious work related to integer sequences [26]. Although this behaviour is not always the case for all consecutive values of d, empirical evidences suggest that more stability is a general tendency of $Q_{d,\ell}(n)$ for ascending d.

Fig. 4. Line graphs of $S_{59,4}(n)$ and $S_{60,4}(n)$ for $n \leq 10^4$.



Our second purpose is understanding the behaviour of $Q_{d,\ell}(n)$ where $d = \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$. Section 2 clearly shows that there is a breaking point for finiteness of $Q_{d,\ell}(n)$. Computer search also confirms this stage of experimentation since $Q_{d,\ell}(n)$ is successfully computed up to $5 \cdot 10^7$ without any termination case for all $4 \leq \ell \leq 100$ where $d = \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$.

$$S_{d,\ell}(n) = Q_{d,\ell}(n) - \frac{(\ell-1)}{\ell} \cdot n, \tag{4}$$

$$\sigma_\ell(d)^2 = \left\langle S_{d,\ell}(n)^2 \right\rangle_N - \left\langle S_{d,\ell}(n) \right\rangle_N^2. \tag{5}$$

4 Discussion and Result

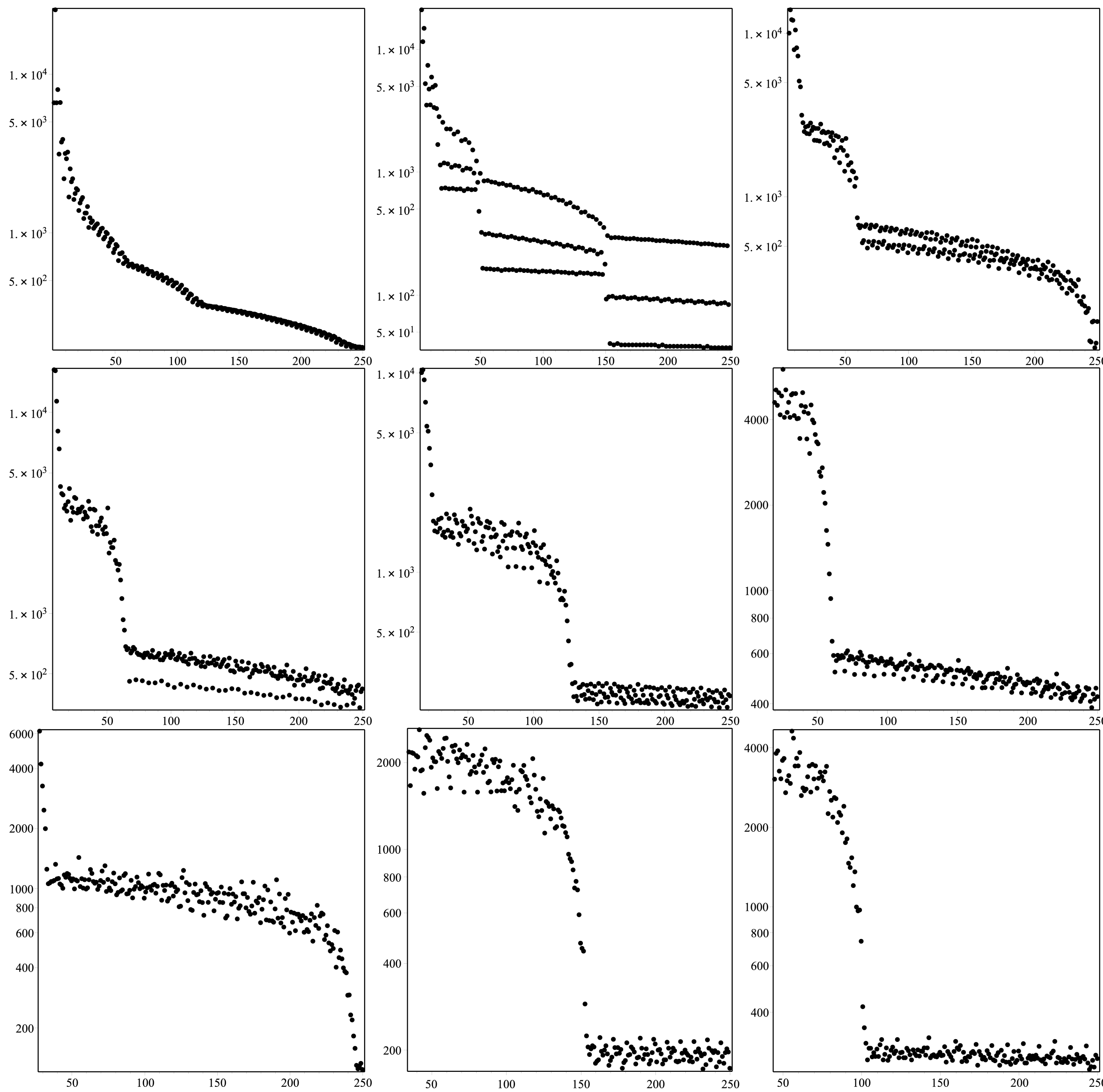
More detailed analysis on these strange recursions can be done thanks to investigation of generational structures of $Q_{d,\ell}(n)$ but this could be extremely diffucult for all chaotic sequences in the range of our experiments since the confirmation of generational boundaries necessitates very careful examination of block structures of each generation for any known generational method [21–23]. On the other hand, Section 2 and Section 3 both fairly conclude that increasing values of $d \geq \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$ tend to stabilize $\sigma_\ell(d)$ levels which are exhibited by oscillations of $Q_{d,\ell}(n)$ around its conjectural slope for sufficiently large fixed N. Additionally, these sections suggest that $Q_{d,\ell}(n)$ tends to be well defined where $d = \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$ for $\ell \geq 4$. So following conjectures are proposed based on reasonable heuristic, analysis and these empirical evidences that previous sections provide.

Conjecture 4.1. *$Q_{d,2}(n)$ is an infinite sequence for all $d > 2$.*

Conjecture 4.2. *There is a m value such that $Q_{d,3}(n)$ is an infinite sequence for all $d > m$.*

Conjecture 4.3. *For any given ℓ, there are infinitely many $d \geq \frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}$ such that $Q_{d,\ell}(n)$ is an infinite sequence.*

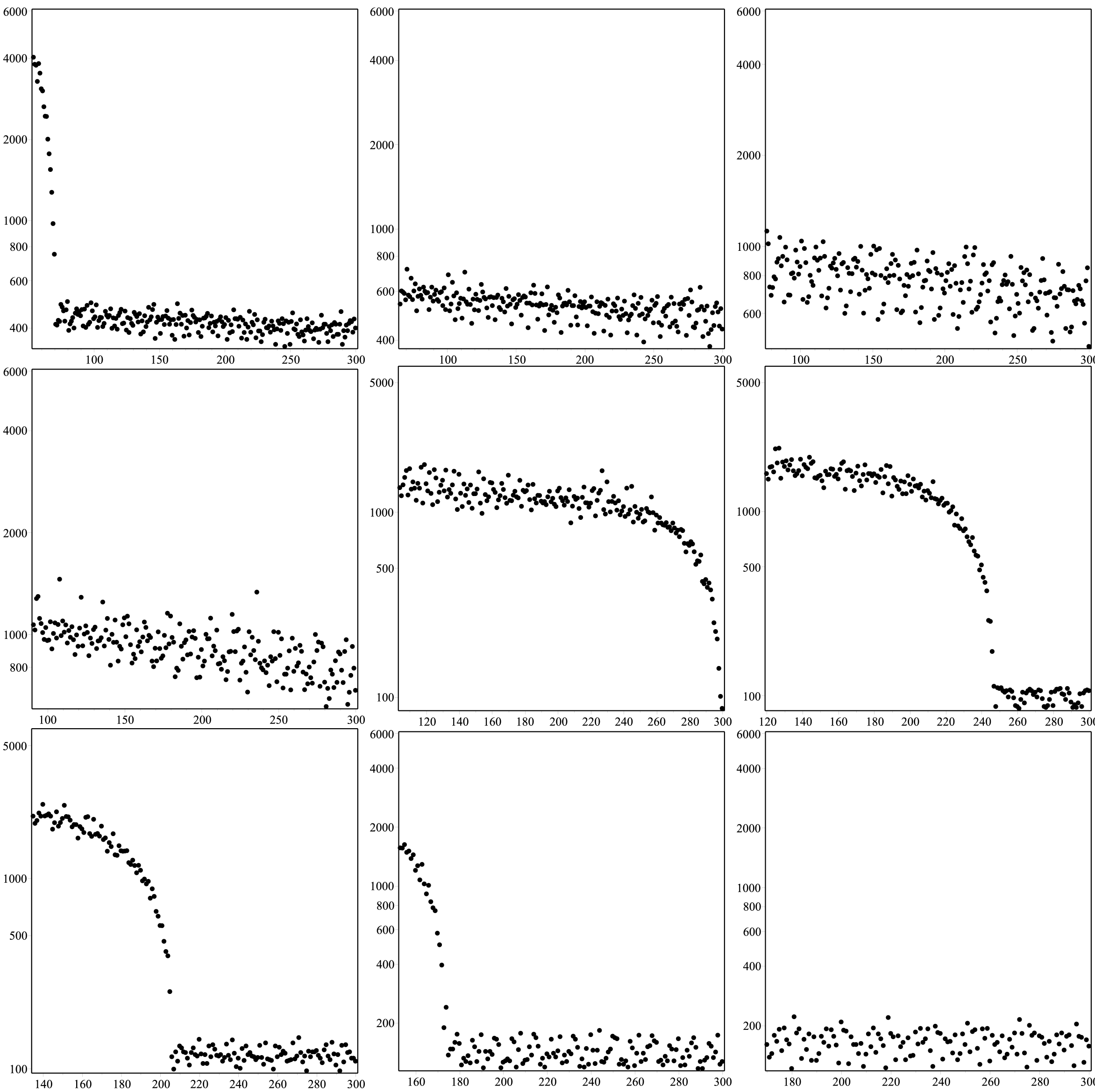
Fig. 5. Logarithmic plots of $\sigma_\ell(d)$ for $\frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2}+0^{|\ell-3|} \le d \le 250$ and $N = 10^6$ where $2 \le \ell \le 10$, respectively.



5 Conclusion

This paper suggests that a generalization $Q_{d,\ell}(n)$ tends to provide coexistence of chaos and order that are recreated by increasing values of d for an arbitrary ℓ. The target of our conjecture is finding a reasonable answer for the most fundamental question about these nested recurrences which this study focuses on. Understanding the behaviors of generalizations of meta-Fibonacci sequences may be really significant and helpful in order to discover curious properties of these kinds of nonlinear recurrences and there can pave the way in this direction due to wealth inheritance of these enigmatic sequences. For example, a new generalization can also be proposed for slow Hofstadter-Conway \$10000 sequence which has curious fractal-like structure [27] as mentioned in A296816. So nature of generalizations of meta-Fibonaci sequences and probable unexpected interactions [22] between curious members of these generalizations contain new enigmatic inquiries for possible future works.

Fig. 6. Logarithmic plots of $\sigma_\ell(d)$ for $\frac{\ell\cdot(\ell-1)-1+(-1)^\ell}{2} \le d \le 300$ and $N = 10^6$ where $11 \le \ell \le 19$, respectively.



Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgement: I would like to thank Prof. Steve Tanny for his helpful communication, Dr. Nathan Fox for his constructive feedback about some of the proofs, and Prof. Robert Israel for his valuable help on Maple-related requirements of this study. Also many thanks to the anonymous referees for making helpful suggestions.

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