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On some classes of 3-dimensional generalized (κ, μ)-contact metric manifolds

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Abstract: The object of the present paper is to obtain a necessary and sufficient condition for a 3-dimensional generalized (κ, μ)-contact metric manifold to be locally ϕ-symmetric in the sense of Takahashi and the condition is verified by an example. Next we characterize a 3-dimensional generalized (κ, μ)-contact metric manifold satisfying certain curvature conditions on the concircular curvature tensor. Finally, we construct an example of a generalized (κ, μ)-contact metric manifold to verify Theorem 1 of our paper.

Key words: Generalized (κ, μ)-contact metric manifolds, concircular curvature tensor, ξ-concircularly flat, locally ϕ-concircularly symmetric

1. Introduction

In a 3-dimensional Riemannian manifold the curvature tensor has a special form. These 3-dimensional Riemannian manifolds have certain characteristic properties that are different from that of a manifold of dimension greater than 3. Due to the interesting properties of 3-dimensional Riemannian manifolds, several authors have studied such manifolds. It is well known that a Sasakian manifold is a normal contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for any $X, Y \in \chi(M)$, where $\chi(M)$ denotes the Lie algebra of all smooth vector fields on M. As a generalization of the Sasakian manifold, Blair et al. [7] introduced the notion of a contact metric manifold called a (κ, μ)-contact metric manifold satisfying the condition

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{1.1}$$

for any $X, Y \in \chi(M)$, where κ and μ are constants on M and $h = \frac{1}{2}L_\xi\phi$ (here L_ξ is the Lie derivative in the direction of ξ). (κ, μ)-contact metric manifolds have been studied by several authors [1, 7, 9, 11, 12, 16, 20, 15].

In 2000, Koufogiorgos and Tsichlias [13] generalized the notion of a (κ, μ)-contact metric manifold by taking the constants κ and μ in (1.1) to be smooth functions on M, called a generalized (κ, μ)-contact metric manifold. Furthermore, the same authors [14] studied 3-dimensional generalized (κ, μ)-contact metric

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manifolds with $\xi\mu = 0$. Three-dimensional generalized (κ,μ)-contact metric manifolds were also studied by De and Ghosh [10] and De and Sarkar [17]. In a recent paper [18], Shaikh et al. proved that a 3-dimensional generalized (κ,μ)-contact metric manifold is locally ϕ-symmetric provided κ and μ are constants. In a recent paper [6], Blair et al. studied concircular curvature tensor in $N(k)$-contact metric manifolds. In 2005, Tripathi and Kim [20] studied the concircular curvature tensor of a (κ,μ)-contact metric manifold.

The present paper is organized as follows: after preliminaries in Section 3, we improve the result of paper [18] and prove that a 3-dimensional generalized (κ,μ)-contact metric manifold is locally ϕ-symmetric if and only if κ and μ are constants. In Section 4, we prove that a ξ-concircularly flat 3-dimensional generalized (κ,μ)-contact metric manifold is either flat or the manifold reduces to a (κ,μ)-contact metric manifold. In the next section, it is shown that a locally ϕ-concircularly symmetric 3-dimensional generalized (κ,μ)-contact metric manifold is a (κ,μ)-contact metric manifold. In Section 6, we study concircularly semisymmetric 3-dimensional generalized (κ,μ)-contact metric manifolds. In the next section, we consider the curvature condition $\widetilde{C}\cdot S = 0$ in a 3-dimensional generalized (κ,μ)-contact metric manifold, where $\widetilde{C}$ is a concircular curvature tensor and S is the Ricci tensor. This result generalizes the results of Tripathi and Kim [20]. Finally, we give an example of a generalized (k,μ)-contact metric manifold to verify the Theorem of Section 3.

2. Preliminaries

In this section, we present some basic facts about contact metric manifolds. We referthe reader to [4] for a more detailed treatment. A differentiable manifold M of dimension $2n+1$ is called a contact manifold if it carries a global 1-form η such that $\eta\wedge(d\eta)^{2n+1}\neq 0$ everywhere on M. The form η is usually called the contact form of M. It is well known that a contact metric manifold admits an almost contact metric structure (ϕ,ξ,η,g), i.e. a global vector field ξ, which is called the characteristic vector field, a $(1,1)$-tensor field ϕ, and a Riemannian metric g such that

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all vector fields $X, Y \in \chi(M)$. Moreover, (ϕ,ξ,η,g) can be chosen such that

$$d\eta(X,Y) = g(X,\phi Y), \qquad X,Y\in\chi(M),$$

and we then call the structure a contact metric structure. A manifold M carrying such a structure is said to be a contact metric manifold and it is denoted by (M,ϕ,ξ,η,g). As a consequence of the above relations, we have $\eta(\xi)=1$, $\phi\xi=0$, $\eta\circ\phi=0$, and $d\eta(\xi,X)=0$. If ∇ denotes the Riemannian connection of (M,ϕ,ξ,η,g), then following [2], we define the $(1,1)$-tensor fields h and l by $h=\frac{1}{2}(L_\xi\phi)$ and $l=R(.,\xi)\xi$, where L_ξ is the Lie differentiation in the direction of ξ and R is the curvature tensor, which is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all vector fields $X,Y,Z\in\chi(M)$. The tensor fields h, l are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad trh = tr\phi h = 0, \quad \phi h + h\phi = 0.$$

Since h anticommutes with ϕ, if $X\neq 0$ is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. Therefore on any contact metric manifold

(M, ϕ, ξ, η, g), the following formulas are valid:

$$\begin{aligned} \nabla \xi &= -\phi - \phi h \ \ \text{(and so } \nabla_\xi \xi = 0\text{)}, \\ \nabla_\xi h &= \phi - \phi l - \phi h^2, \\ \phi l \phi - l &= 2(\phi^2 + h^2). \end{aligned}$$

A contact metric structure (ϕ, ξ, η, g) on M gives rise to an almost complex structure on the product $M \times \Re$. If this structure is integrable, then the contact metric manifold (M, ϕ, ξ, η, g) is said to be Sasakian. Equivalently, a contact metric manifold (M, ϕ, ξ, η, g) is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all $X, Y \in \chi(M)$.

By a generalized (κ, μ)-manifold, we mean a 3-dimensional contact metric manifold such that

$$R(X,Y)\xi = \{\kappa I + \mu h\}[\eta(Y)X - \eta(X)Y], \tag{2.1}$$

for all $X, Y \in \chi(M)$, where κ, μ are smooth nonconstant real functions on M. In the special case where κ, μ are constant, then (M, ϕ, ξ, η, g) is called a (κ, μ)-manifold. We note that in any Sasakian manifold $h = 0$ and $\kappa = 1$.

Lemma 1 *[13, 14] In any generalized (κ, μ)-manifold (M, ϕ, ξ, η, g) the following formulas are valid :*

$$\begin{aligned} h^2 &= (\kappa - 1)\phi^2, \quad \kappa = \frac{trl}{2} \leq 1, \\ \xi\kappa &= 0, \ \ \xi r = 0, \\ hgrad\mu &= grad\kappa. \end{aligned}$$

Lemma 2 *[3] A $(2n+1)$-dimensional contact metric manifold satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat if $n = 1$.*

Lemma 3 *[13] Let M be a non-Sasakian, generalized (κ, μ)-contact metric manifold. If κ, μ satisfy the condition $a\kappa + b\mu = c$ (a, b, c are constants), then κ, μ are constants.*

Lemma 4 *[5] A 3-dimensional contact metric manifold $(M^3, \phi, \xi, \eta, g)$ with $Q\phi = \phi Q$ is either Sasakian, flat, or of constant ξ-sectional curvature $\kappa < 1$ and constant ϕ-sectional curvature $-\kappa$.*

Furthermore, in a generalized (κ, μ)-contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we have the following [2]:

$$\begin{aligned} (\nabla_X h)Y &= \{(1-\kappa)g(X, \phi Y) - g(X, \phi hY)\}\xi - \eta(Y)\{(1-\kappa)\phi X + \phi hX\} \\ &\quad -\mu\eta(X)\phi hY, \end{aligned} \tag{2.2}$$

$$(\nabla_X \phi)Y = \{g(X,Y) + g(X, hY)\}\xi - \eta(Y)(X + hX). \tag{2.3}$$

For a 3-dimensional Riemannian manifold, the conformal curvature tensor $C = 0$. Thus, the curvature tensor R and Ricci tensor S for a 3-dimensional generalized (κ, μ)-contact metric manifold are given by [18]:

$$\begin{aligned} R(X,Y)Z &= -(\kappa+\mu)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad +(2\kappa+\mu)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad +\mu\{g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y\}, \end{aligned} \tag{2.4}$$

$$S(X,Y) = -\mu g(X,Y) + \mu g(hX,Y) + (2\kappa+\mu)\eta(X)\eta(Y), \tag{2.5}$$

respectively. We see that on a 3-dimensional generalized (κ, μ)-contact metric manifold the scalar curvature r is equal to

$$r = 2(\kappa - \mu). \tag{2.6}$$

In an n-dimensional Riemannian manifold, the concircular curvature tensor $\widetilde{C}$ is defined by [21, 22]

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.7}$$

We observe from (2.7) that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature is a measure of the failure of a Riemannian manifold to be of constant curvature.

Hence, using (2.4) and (2.6) in (2.7), we get

$$\begin{aligned} \widetilde{C}(X,Y)Z &= -(\frac{4\kappa+2\mu}{3})\{g(Y,Z)X - g(X,Z)Y\} + (2\kappa+\mu)\{g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad +\mu\{g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y\}. \end{aligned} \tag{2.8}$$

Using (2.8), we can give the following formulas:

$$\widetilde{C}(X,Y)\xi = \frac{2\kappa+\mu}{3}\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.9}$$

$$\widetilde{C}(X,\xi)Z = \frac{2\kappa+\mu}{3}\{\eta(Z)X - g(X,Z)\xi\} + \mu\{\eta(Z)hX - g(hX,Z)\xi\}, \tag{2.10}$$

$$\widetilde{C}(X,\xi)\xi = \frac{2\kappa+\mu}{3}\{X - \eta(X)\xi\} + \mu hX, \tag{2.11}$$

$$S(X,\xi) = 2\kappa\eta(X). \tag{2.12}$$

3. Locally ϕ-symmetric 3-dimensional generalized (κ, μ)-contact metric manifolds

Definition 1 *A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be locally ϕ-symmetric in the sense of Takahashi [19] if it satisfies*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ.

In a 3-dimensional generalized (κ,μ)-contact metric manifold, if we take X,Y,Z,W to be horizontal vector fields, that is, if X,Y,Z,W are orthogonal to ξ, then we obtain [18]

$$\begin{aligned}\phi^2(\nabla_W R)(X,Y)Z &= (W\kappa+W\mu)\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad -(W\mu)\{g(Y,Z)hX-g(X,Z)hY\\ &\quad +g(hY,Z)X-g(hX,Z)Y\}. \end{aligned} \tag{3.1}$$

Suppose that κ and μ are constants; then a 3-dimensional generalied (κ,μ)-contact metric manifold is locally ϕ-symmetric in the sense of Takahashi. Conversely, let the manifold under consideration be locally ϕ-symmetric. Then, from (3.1), we obtain

$$\begin{aligned}&(W\kappa+W\mu)\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}\\ &-(W\mu)\{g(Y,Z)g(hX,U)-g(X,Z)g(hY,U)+g(hY,Z)g(X,U)\\ &-g(hX,Z)g(Y,U)\}=0.\end{aligned} \tag{3.2}$$

Contracting X and Z in (3.2), we obtain

$$2n(W\kappa+W\mu)g(Y,U)+(2n-1)(W\mu)g(Y,hU)=0, \tag{3.3}$$

which implies

$$2n(W\kappa+W\mu)Y+(2n-1)(W\mu)hY=0. \tag{3.4}$$

Applying h in (3.4) and taking the trace of h we get

$$(W\mu)=0, \tag{3.5}$$

since trace $h^2\neq 0$.
That is, $\mu=$constant. Using $\mu=$constant in (3.4), we get $\kappa=$ constant. Thus, we can state the following:

Theorem 1 *A 3-dimensional generalized (κ,μ)-contact metric manifold is locally ϕ-symmetric if and only if κ and μ are constants.*

Now we obtain the following:

Corollary 1 *A 3-dimensional (κ,μ)-contact metric manifold is always locally ϕ-symmetric.*

4. ξ-concircularly flat 3-dimensional (κ,μ)-contact metric manifolds

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g). At each point $p\in M$, decompose the tangent space T_pM into direct sum $T_pM=\phi(T_pM)\oplus\{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus, the conformal curvature tensor C is a map

$$C: T_pM\times T_pM\times T_pM\longrightarrow \phi(T_pM)\oplus\{\xi_p\},\ p\in M.$$

It may be natural to consider the following particular cases:

1. $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow L(\xi_p)$, i.e. the projection of the image of C in $\phi(T_pM)$ is zero.

2. $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \phi(T_p(M))$, i.e. the projection of the image of C in $L\xi_p$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0. \tag{4.1}$$

3. $C : \phi(T_p(M))\times\phi(T_p(M))\times\phi(T_p(M)) \longrightarrow L(\xi_p)$, i.e. when C is restricted to $\phi(T_pM)\times\phi(T_pM)\times\phi(T_pM)$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{4.2}$$

A Riemannian manifold satisfying (4.1) is called ξ-conformally flat. The Riemannian manifold satisfying cases (4.1) and (4.2) was considered in [23, 24, 8], respectively.

Analogous to the consideration of a conformal curvature tensor, here we can define the following:

Definition 2 *A $(2n+1)$-dimensional contact metric manifold is said to be ξ-concircularly flat if*

$$\widetilde{C}(X,Y)\xi = 0. \tag{4.3}$$

In this section we prove the following:

Theorem 2 *A ξ-concircularly flat 3-dimensional non-Sasakian generalized (κ,μ)-contact metric manifold is either flat or reduces to a (κ,μ) contact metric manifold.*

Proof For a ξ-concircularly flat manifold from (2.9), we get

$$\frac{2\kappa+\mu}{3}\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} = 0. \tag{4.4}$$

If we take $X = \xi$ in (4.4), we obtain

$$\frac{2\kappa+\mu}{3}\{\eta(Y)\xi - Y\} - \mu hY = 0. \tag{4.5}$$

Taking $Y = hY$ in (4.5), we have

$$-\frac{2\kappa+\mu}{3}hY - \mu h^2 Y = 0. \tag{4.6}$$

Taking the trace of h we get $\mu = 0$. Using $\mu = 0$ in (4.4) we get $\kappa = 0$. Thus, $R(X,Y)\xi = 0$, and hence, from Lemma 2, we get that the manifold is flat. Taking the inner product with U in the equation (4.4) and contracting X and U, we obtain $2\kappa+\mu = 0$. Applying Lemma 3, we get $k, \mu =$ constant. Thus, the manifold reduces to a (k,μ)-contact metric manifold. □

5. Locally ϕ-concircularly symmetric 3-dimensional generalized (κ,μ)-contact metric manifolds

In this section, first we give the following:

Definition 3 *A contact metric manifold M is said to be locally ϕ-concircularly symmetric in the sense of Takahashi [19] if it satisfies*

$$\phi^2((\nabla_W \widetilde{C})(X,Y)Z) = 0. \tag{5.1}$$

Now we investigate locally ϕ-concircularly symmetric 3-dimensional generalized (κ,μ)-contact metric manifolds. First we can write the following:

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)Z &= \nabla_W \widetilde{C}(X,Y)Z - \widetilde{C}(\nabla_W X,Y)Z \\ &\quad -\widetilde{C}(X,\nabla_W Y)Z - \widetilde{C}(X,Y)\nabla_W Z.\end{aligned} \tag{5.2}$$

Using (2.8) in (5.2), we obtain

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)Z &= -\frac{1}{3}((W\kappa)+(W\mu))\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad +(2(W\kappa)+(W\mu))\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad +(W\mu)\{g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y\} \\ &\quad +(2\kappa+\mu)\{g(Y,Z)g(X,\nabla_W\xi)\xi + g(Y,Z)\eta(X)\nabla_W\xi \\ &\quad -g(X,Z)g(Y,\nabla_W\xi)\xi - g(X,Z)\eta(X)\xi + g(Y,\nabla_W\xi)\eta(Z)X \\ &\quad +\eta(Y)g(Z,\nabla_W\xi)X - g(X,\nabla_W\xi)\eta(Z)Y - \eta(X)g(Z,\nabla_W\xi)Y\} \\ &\quad +\mu\{g(Y,Z)(\nabla_W h)X + g(((\nabla_W h)Y,Z)X \\ &\quad -g((\nabla_W h)X,Z)Y - g(X,Z)(\nabla_W h)Y\}.\end{aligned} \tag{5.3}$$

Applying ϕ^2 to both sides of (5.3) and taking X, Y, Z, and W as horizontal vector fields, we get

$$\begin{aligned}&\frac{1}{3}(4(W\kappa)+2(W\mu))\{g(Y,Z)X - g(X,Z)Y\} \\ &+(W\mu)\{g(X,Z)hY - g(Y,Z)hX - g(hY,Z)X + g(hX,Z)Y\} = 0.\end{aligned} \tag{5.4}$$

Applying h to (5.4) and taking the trace of h, we have

$$(W\mu)\{g(X,Z)traceh^2Y - g(Y,Z)traceh^2X\} = 0. \tag{5.5}$$

Contracting X and Z in (5.5), we obtain

$$2n(W\mu)traceh^2Y = 0. \tag{5.6}$$

Since $traceh^2 \neq 0$, from (5.6) it follows that $\mu =$constant. Using $\mu =$constant in (5.4) yields $\kappa =$constant. Hence, the manifold reduces to a (κ,μ)-contact metric manifold. Thus, we can give the following:

Theorem 3 *A locally ϕ-concircularly symmetric 3-dimensional generalized (κ,μ)-contact metric manifold reduces to a (κ,μ)-contact metric manifold.*

6. Three-dimensional generalized (κ,μ)-contact metric manifolds satisfying the condition $R\cdot\widetilde{C}=0$

Let M be a 3-dimensional generalized (κ,μ)-contact metric manifold satisfying the condition $R\cdot\widetilde{C}=0$. Then we can write

$$\begin{aligned}(R(X,Y)\cdot\widetilde{C})(U,V)W &= R(X,Y)\widetilde{C}(U,V)W-\widetilde{C}(R(X,Y)U,V)W\\ &\quad -\widetilde{C}(U,R(X,Y)V)W-\widetilde{C}(U,V)R(X,Y)W=0. \qquad (6.1)\end{aligned}$$

If we take $X=V=\xi$ in (6.1), we get

$$\begin{aligned}(R(\xi,Y)\cdot\widetilde{C}(U,\xi)W &= R(\xi,Y)\widetilde{C}(U,\xi)W-\widetilde{C}(R(\xi,Y)U,\xi)W \qquad (6.2)\\ &\quad -\widetilde{C}(U,R(\xi,\xi)V)W-\widetilde{C}(U,\xi)R(\xi,Y)W=0.\end{aligned}$$

Using (2.4) in (6.2), we have

$$\begin{aligned}&\kappa g(Y,\widetilde{C}(U,\xi)W)\xi-\kappa\eta(\widetilde{C}(U,\xi)W)Y+\mu g(hY,\widetilde{C}(U,\xi)W)\xi\\ &-\mu\eta(\widetilde{C}(U,\xi)W)hY-\kappa g(Y,U)\widetilde{C}(\xi,\xi)W+\kappa\eta(U)\widetilde{C}(Y,\xi)W\\ &-\mu g(hY,U)\widetilde{C}(\xi,\xi)W+\mu\eta(U)\widetilde{C}(hY,\xi)W-\kappa\eta(Y)\widetilde{C}(U,\xi)W \qquad (6.3)\\ &+\kappa\widetilde{C}(U,Y)W+\mu\widetilde{C}(U,hY)W=0.\end{aligned}$$

Using (2.10) in (6.3), we write

$$\begin{aligned}&\kappa(\frac{2\kappa+\mu}{3})\{\eta(W)g(Y,U)\xi-g(U,W)\eta(Y)\xi-g(Y,W)\eta(U)\xi\\ &-\eta(Y)\eta(W)U-\eta(Y)g(U,W)\xi\}\\ &+\kappa\mu\{\eta(W)g(hY,U)\xi-\eta(U)g(hY,W)\xi+\eta(U)\eta(W)hY\\ &-\eta(U)g(hY,W)\xi-\eta(Y)\eta(W)hU+g(Y,W)hU-g(U,W)hY+g(hY,W)U\}\\ &-\mu(\frac{4\kappa+2\mu}{3})\{g(hY,W)U-g(U,W)hY\}+\mu^2\{\eta(W)g(hY,hU)\xi\\ &+\eta(U)\eta(W)h^2Y-\eta(U)g(h^2Y,W)\xi+g(hY,W)hU \qquad (6.4)\\ &-g(U,W)h^2Y-g(h^2Y,W)U\}\\ &-\kappa(\frac{4\kappa+2\mu}{3})\{g(Y,W)U-g(U,W)Y\}+\kappa(2\kappa+\mu)\{g(Y,W)\eta(U)\xi\\ &-g(U,W)\eta(Y)\xi+\eta(Y)\eta(W)U-\eta(U)\eta(W)hY\}\\ &+\mu(\frac{2\kappa+\mu}{3})\{\eta(W)g(hY,U)\xi-g(hY,W)\xi\}\\ &+\mu(2\kappa+\mu)\{g(hY,W)\eta(U)\xi-\eta(U)\eta(W)hY\}=0\end{aligned}$$

Taking the inner product with T and contracting Y and T in (6.4), we obtain

$$\begin{aligned}&\{5\kappa\frac{2\kappa+\mu}{3}+\mu^2 g(h^2e_i,e_i)\}(\eta(U)\eta(W)-g(W,U)) \qquad (6.5)\\ &+\{\kappa\mu-\mu(\frac{4\kappa+2\mu}{3})\}g(hW,U)=0.\end{aligned}$$

Instead of (6.5), we can write

$$\{5\kappa\frac{2\kappa+\mu}{3}+\mu^2 g(h^2 e_i,e_i)\}(\eta(U)\xi-U)$$
$$+\{\kappa\mu-\mu(\frac{4\kappa+2\mu}{3})\}hU=0. \qquad (6.6)$$

Operating h and taking the trace of h, we obtain $\kappa\mu-\mu(\frac{4\kappa+2\mu}{3})=0$. Therefore, either $\mu=0$ or $\kappa+2\mu=0$. By Lemma 3, we get from $\kappa+2\mu=0$ that the manifold is a (κ,μ)-contact manifold. We also know $Q\phi-\phi Q=2\mu h\phi$. Therefore, $\mu=0$ implies $Q\phi=\phi Q$. Then by Lemma 4, we get that the manifold is either Sasakian, flat, or of constant ξ-sectional curvature $\kappa<1$ and constant ϕ-sectional curvature -1.

Hence, we can give the following:

Theorem 4 *A 3-dimensional non-Sasakian generalized (κ,μ)-contact metric manifold satisfying the condition $R\cdot\widetilde{C}=0$ is either a (κ,μ)-contact manifold, flat, or of constant ξ-sectional curvature $\kappa<1$ and constant ϕ-sectional curvature -1.*

7. Three-dimensional generalized (κ,μ)-contact metric manifolds satisfying the condition $\tilde{C}\cdot S=0$

Let M be a 3-dimensional generalized (κ,μ)-contact metric manifold satisfying the condition $\widetilde{C}(\xi,X)\cdot S=0$. Then we can write

$$(\widetilde{C}(\xi,X)\cdot S)(Y,W)=-S(\widetilde{C}(\xi,X)Y,W)-S(Y,\widetilde{C}(\xi,X)W)=0,$$

or equivalently,

$$S(\widetilde{C}(\xi,X)Y,W)+S(Y,\widetilde{C}(\xi,X)W)=0. \qquad (7.1)$$

Using (2.10) in (7.1), we get

$$-\mu\frac{2\kappa+\mu}{3}\{g(X,Y)\eta(W)-g(X,W)\eta(Y)-\eta(Y)g(hX,W)\}$$
$$+\mu^2\{\eta(Y)g(hX,W)-g(hX,Y)\eta(W)\}$$
$$+\frac{(2\kappa+\mu)^2}{3}\{g(X,Y)\eta(W)-\eta(X)\eta(W)\eta(Y)\}+\mu(2\kappa+\mu)g(hX,Y)\eta(W)$$
$$+\mu^2(k-1)g(X,W)\eta(Y)-\mu_2(k-1)\eta(Y)\eta(W)\eta(X)-\mu\frac{2\kappa+\mu}{3}\{g(X,W)\eta(Y) \qquad (7.2)$$
$$-g(X,Y)\eta(W)+\eta(W)g(hY,X)\}+\mu^2\{\eta(W)g(hX,Y)-g(hX,W)\eta(Y)\}$$
$$+\{g(X,W)\eta(Y)-\eta(W)\eta(Y)\eta(X)\}+\mu(2\kappa+\mu)g(hX,W)\eta(Y)$$
$$+\mu^2(k-1)g(X,Y)\eta(W)-\mu_2(k-1)\eta(Y)\eta(W)\eta(X)=0,$$

or equivalently,

$$\mu\frac{2\kappa+\mu}{3}\{\eta(Y)g(hX,W)+\eta(W)g(hY,X)\eta(W)\}$$
$$-\mu(2\kappa+\mu)\{g(hX,Y)\eta(W)+g(hX,W)\eta(Y)\}+\frac{(2\kappa+\mu)^2}{3} \qquad (7.3)$$
$$+\mu^2(\kappa-1))\{g(X,Y)\eta(W)-2\eta(X)\eta(Y)\eta(W)+g(X,W)\eta(Y)\}=0.$$

Taking $W=\xi$ in (7.3), we obtain

$$2\mu\frac{2\kappa+\mu}{3}g(hY,X)+(\frac{(2\kappa+\mu)^2}{3}+\mu^2(\kappa-1))\{g(X,Y)-2\eta(X)\eta(Y)+\eta(X)\eta(Y)\}=0. \tag{7.4}$$

Equation (7.4) can be written as

$$2\mu\frac{2\kappa+\mu}{3}hY+(\frac{(2\kappa+\mu)^2}{3}+\mu^2(\kappa-1))\{Y-\eta(Y)\xi\}=0. \tag{7.5}$$

Operating h in (7.5), we get

$$2\mu\frac{2\kappa+\mu}{3}h^2Y+(\frac{(2\kappa+\mu)^2}{3}+\mu^2(\kappa-1))hY=0. \tag{7.6}$$

Taking the trace of h and using $trh=0$ in (7.6), we get $\mu(2\kappa+\mu)=0$, since $trh^2\neq 0$. Therefore, either $\mu=0$ or $2\kappa+\mu=0$. By Lemma 3, we get from $2\kappa+\mu=0$ that the manifold becomes a (κ,μ)-contact metric manifold. Also, if $\mu=0$, then by Lemma 4, we get that the manifold is either Sasakian, flat, or of constant ξ-sectional curvature $\kappa<1$ and constant ϕ-sectional curvature -1.

Thus, we are in a position to state the following:

Theorem 5 *A 3-dimensional generalized (κ,μ)-contact metric manifold satisfying the condition $\widetilde{C}\cdot S=0$ is either Sasakian, flat, or of constant ξ-sectional curvature $\kappa<1$ and constant ϕ-sectional curvature -1, or a (κ,μ)-contact metric manifold.*

8. An example of generalized (κ,μ)-contact metric manifolds

Let $k:I\subset\mathbb{R}\to\mathbb{R}$ be a smooth function defined on an open interval I, such that $k(z)<1$ for any $z\in I$. Then we construct a generalized (κ,μ)-contact metric manifold (M,ϕ,ξ,η,g) in the set $M=\mathbb{R}^2\times I\subset\mathbb{R}^3$ as follows:

We put $\lambda(z)=\sqrt{1-k(z)}>0$, $\lambda'(z)=\frac{\partial\lambda}{\partial z}$ and we consider three linearly independentvector fields e_1,e_2, and e_3 on M as follows:

$$\begin{aligned}
e_1 &= \frac{\partial}{\partial x},\quad e_2=\frac{\partial}{\partial y},\\
e_3 &= [2y+f(z)]\frac{\partial}{\partial x}+[2\lambda(z)x-\frac{\lambda'(z)}{2\lambda(z)}y+h(z)]\frac{\partial}{\partial y}+\frac{\partial}{\partial z},
\end{aligned}$$

where $f(z)$ and $h(z)$ are arbitrary functions of z. We define the tensor fields ϕ, η, g as follows:

g is the Riemannian metric on M with respect to which the vector fields e_1,e_2,e_3 are orthonormal; η is the 1-form on M defined by $\eta(Z)=g(Z,e_1)$ for any $Z\in\chi(M)$; ϕ is the $(1,1)$-tensor field defined by $\phi e_1=0,\phi e_2=e_3,\phi e_3=-e_2$.

Now we calculate the following:

$$[e_1,e_2]=0,\ \ [e_1,e_3]=2\lambda(z)e_2,\ \ [e_2,e_3]=-\frac{\lambda'(z)}{2\lambda(z)}e_2+2e_1.$$

Since $(\eta \wedge d\eta)(e_1, e_2, e_3) \neq 0$ everywhere on M, we conclude that η is a contact form. From the definition of ϕ, g and the relations of (8.1) it is easy to verify that

$$\begin{aligned} \phi^2 Z &= -Z + \eta(Z)e_1, \\ g(\phi Y, \phi Z) &= g(Y,Z) - \eta(Y)\eta(Z), \\ d\eta(Y,Z) &= g(Y, \phi Z), \end{aligned}$$

for any $Y, Z \in \chi(M)$. Therefore, (M, ϕ, ξ, η, g) is a contact metric manifold for $\xi = e_1$. Using Koszul's formula we calculate the following:

$$\begin{aligned} \nabla_{e_1} e_1 &= 0 \ \nabla_{e_1} e_2 = -\{1+\lambda(z)\}e_3, \ \ \nabla_{e_1} e_3 = \{1+\lambda(z)\}e_2 \\ \nabla_{e_2} e_1 &= -\{1+\lambda(z)\}e_3, \ \ \nabla_{e_2} e_3 = \frac{\lambda'(z)}{2\lambda(z)} e_3, \ \ \nabla_{e_2} e_3 = -\frac{\lambda'(z)}{2\lambda(z)} e_2 + \{1+\lambda(z)\}e_1 \end{aligned}$$

$$\nabla_{e_3} e_1 = \{1-\lambda(z)\}e_2, \ \ \nabla_{e_3} e_2 = -\{1-\lambda(z)\}e_1, \ \ \nabla_{e_3} e_3 = 0. \tag{8.1}$$

Comparing with the relation $\nabla_X e_1 = -\phi X - \phi h X$ and the relations obtained in (8.1), we see that

$$he_1 = 0, \ \ he_2 = \lambda(z)e_2, \ \ he_3 = -\lambda(z)e_3.$$

Using the formula $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we calculate the following:

$$\begin{aligned} R(e_2, e_1)e_1 &= \{1+\lambda(z)\}^2 e_2 \\ &= \{1 + 2\lambda(z) + (\lambda(z))^2\}e_2 \\ &= [\{1 - (\lambda(z))^2\} + 2\{1+\lambda(z)\}\lambda(z)]e_2 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_1)e_2 - \eta(e_2)e_1) + 2\{1+\lambda(z)\}(\eta(e_1)he_2 - \eta(e_2)he_1), \end{aligned}$$

$$\begin{aligned} R(e_3, e_1)e_1 &= \{1 - 2\lambda(z) - 3(\lambda(z))^2\}e_3 \\ &= [\{1 - (\lambda(z))^2\} + 2\{1+\lambda(z)\}(-\lambda(z))]e_3 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_1)e_3 - \eta(e_3)e_1) + 2\{1+\lambda(z)\}(\eta(e_1)he_3 - \eta(e_3)he_1), \end{aligned}$$

$$\begin{aligned} R(e_2, e_3)e_1 &= 0 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_3)e_2 - \eta(e_2)e_3) + 2\{1+\lambda(z)\}(\eta(e_3)he_2 - \eta(e_2)he_3). \end{aligned}$$

In view of the above expressions of curvature tensors we can easily conclude that the manifold M is a generalized (κ, μ)-contact metric manifold with $k = \{1 - (\lambda(z))^2\}$ and $\mu = 2\{1+\lambda(z)\}$.

Other curvature tensors are given below:

$$\begin{aligned}
R(e_1,e_2)e_3 &= 0,\\
R(e_1,e_2)e_2 &= \{1+\lambda(z)\}e_1,\\
R(e_1,e_3)e_3 &= \{1-2\lambda(z)-3(\lambda(z))^2\}e_1,\\
R(e_3,e_2)e_2 &= \{\frac{\lambda''(z)}{2(\lambda(z))^2}-\frac{3(\lambda'(z))^2}{4(\lambda(z))^2}-2\{1+\lambda(z)\}-\{1-(\lambda(z))^2\}\}e_3\\
R(e_2,e_3)e_3 &= \{\frac{\lambda''(z)}{2(\lambda(z))^2}-\frac{3(\lambda'(z))^2}{4(\lambda(z))^2}-2\{1+\lambda(z)\}-\{1-(\lambda(z))^2\}\}e_2.
\end{aligned}$$

If we consider $\lambda(z) = \text{constant} > 0$, then κ and μ become constants and hence the manifold becomes a (κ,μ)-contact metric manifold.

It can be easily verified that such a (κ,μ)-contact metric manifold is locally ϕ-symmetric. Then Theorem 1 is verified.

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