J. Math. Anal. Appl. 430 (2015) 549–571

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Identifying weak foci and centers in the Maxwell–Bloch system

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ARTICLE INFO

*Article history:*
Received 23 July 2014
Available online 7 May 2015
Submitted by J. Shi

*Keywords:*
Maxwell–Bloch system
Center-focus
Algebraic variety
Invariant algebraic surface
Singular center

ABSTRACT

In this paper we identify weak foci and centers in the Maxwell–Bloch system, a three dimensional quadratic system whose three equilibria are all possible to be of center-focus type. Applying irreducible decomposition and the isolation of real roots in computation of algebraic varieties of Lyapunov quantities on an approximated center manifold, we prove that at most 6 limit cycles arise from Hopf bifurcations and give conditions for exact number of limit cycles near each weak focus. Further, applying algorithms of computational commutative algebra we find Darboux polynomials and give some center manifolds in closed form globally, on which we identify equilibria to be centers or singular centers by integrability and time-reversibility on a center manifold. We prove that those centers are of at most second order.


## 1. Introduction

The description of the interaction between laser light and a material sample composed of two-level atoms begins with Maxwell equations of the electric field and Schrödinger equations for the probability amplitudes of the atomic levels [10,16]. In 1985, coupling the Maxwell equations with the Bloch equation (a linear Schrödinger like equation describing the evolution of atoms resonantly coupled to the laser field), F.T. Arecchi [1] proposed the 3-dimensional quadratic differential system

$$\begin{cases} \dot{E} = -k\,E + g\,P, \\ \dot{P} = -\gamma_\perp\,P + g\,E\Delta, \\ \dot{\Delta} = -\gamma_\parallel\,(\Delta - \delta_0) - 4g\,PE, \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2015.05.007

called the Maxwell–Bloch system, where $E$ is the coupling of the fundamental cavity mode, $P$ is the collective atomic polarization, $\Delta$ represents the population inversion, $k$, $\gamma_\perp$, $\gamma_\|$ are the loss rates of field, polarization and population respectively, $g$ is a coupling constant ($g \neq 0$) and $\delta_0$ is the population inversion which would be established by the pump mechanism in the atomic medium, in the absence of coupling. As indicated in [1,15,17], the Maxwell–Bloch system describes Type I laser (He-Ne), Type II laser (Ruby and $CO_2$) and Type III laser (far infrared) when $\gamma_\perp \approx \gamma_\| \gg k$, when $\gamma_\perp \gg \gamma_\| \approx k$ and when $\Delta_0$ is large enough, respectively. Numerical simulations [1] show that Type I laser and Type II laser are both stable but Type III laser is unstable.

A few papers [15,23] were published to discuss this 3-dimensional system in mathematical aspect. In 1998 Puta [23] considered the integrability in the special case that $k = \gamma_\perp = \gamma_\| = 0$. In 2010 Hacinliyan, Kusbeyzi and Aybar [15] indicated that the pair of equilibria which are $\mathcal{C}_2$-symmetric with respect to the $\Delta$-axis are both of center-focus type and showed the rise of a stable limit cycle from a Hopf bifurcation, verifying that a pair of conjugate complex eigenvalues crosses the imaginary axis and varying the parameter $g$ to obtain a negative first Lyapunov coefficient numerically. However, the work of [15] neither answers the order of weak foci nor identifies a center. Actually, it causes great complexity in computation to identify weak foci and centers for 3-dimensional systems [9,14,19–21]. Concerning weak foci, a natural idea is to compute Lyapunov quantities for the restriction of the 3-dimensional system to an approximated center manifold, but the approximation to the center manifold makes the complicated computation of algebraic varieties of Lyapunov quantities more difficult. Concerning centers, those criteria for centers in planar systems, seen in [22,24] for time-reversibility and integrability, are not effective on an approximated center manifold. In [20] such identifications for weak focus and center were completed for the generalized Lorenz system, a 3-dimensional system, by approximating a local center manifold and finding the closed form of a global center manifold respectively. However, system (1.1) is quite different because it is only for $\delta_0 = (k+1)(\gamma_\perp+1)$ that system (1.1) can be transformed into the generalized Lorenz form.

In this paper we identify weak foci and centers for the Maxwell–Bloch system (1.1) qualitatively. With a time rescaling $\tau_1 = gt$, system (1.1) can be simplified as the following equivalent form

$$\begin{cases} \dot{x}_1 = -a\,x_1 + x_2, \\ \dot{x}_2 = -b\,x_2 + x_1x_3, \\ \dot{x}_3 = -c\,(x_3 - \delta_0) - 4\,x_1x_2, \end{cases} \tag{1.2}$$

where $x_1$, $x_2$, $x_3$ simply present $E$, $P$, $\Delta$ respectively and $a = g^{-1}k$, $b = g^{-1}\gamma_\perp$, $c = g^{-1}\gamma_\|$, the ratios of the loss rates of field $k$, polarization $\gamma_\perp$ and population $\gamma_\|$ respectively to the coupling constant. The three equilibria $E_0$ and $E_\pm$ are all possible to be of center-focus type. We prove that the system can totally produce at most 6 limit cycles from those weak foci. Applying irreducible decomposition and the isolation of real roots in computation of algebraic varieties of Lyapunov quantities on approximated center manifolds, we give conditions for exact number of limit cycles near each weak focus. Further, applying algorithms of computational commutative algebra, we find Darboux factors in polynomial form, which give some center manifolds in closed form globally. We identify equilibria to be centers or singular centers by proving integrability and time-reversibility on a center manifold. We prove that $E_\pm$ are both rough centers but $E_0$ is a center of at most order two.

## 2. Weak foci

As known in [15], the authors give the qualitative properties of all equilibria for system (1.1). The reduced system (1.2) containing less parameters helps us in latter computation of center manifolds, normal forms and those determining quantities. Clearly, for $c = 0$ system (1.2) has a singular line $\{(x_1, x_2, x_3) | x_1 = 0, x_2 = 0\}$.

For $c \neq 0$, system (1.2) has exactly one equilibrium $E_0 : (0, 0, \delta_0)$ when $ac(\delta_0 - ab) \leq 0$ but three equilibria $E_0$, $E_+ : (x_1^*, x_2^*, x_3^*)$ and $E_- : (-x_1^*, -x_2^*, x_3^*)$ where

$$x_1^* = \sqrt{\frac{c(\delta_0 - ab)}{4a}}, \quad x_2^* = a\sqrt{\frac{c(\delta_0 - ab)}{4a}}, \quad x_3^* = ab$$

when $ac(\delta_0 - ab) > 0$.

The Jacobian matrix of system (1.2) at the equilibrium $E_0$ is

$$J(E_0) = \begin{bmatrix} -a & 1 & 0 \\ \delta_0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix},$$

which has eigenvalues $-c$ and $-\frac{1}{2}(a+b) \pm \frac{1}{2}\sqrt{(a-b)^2 + 4\delta_0}$. Then, $E_0$ is a locally stable node or focus in the case that $\delta_0 < ab$, $a + b > 0$, $c > 0$. Notice that Type I laser (He-Ne) and Type II laser (Ruby and $CO_2$) were illustrated in [1] for $\gamma_\perp = \gamma_\| = 10^9$, $k = 10^7$ and $\gamma_\perp = 10^8$, $\gamma_\| = k = 2.5$ respectively. We can check that the parameter conditions of Type I and II lasers satisfy that $\delta_0 < ab$, $a + b > 0$, $c > 0$ for $\delta_0$ is small. It implies that $E_0$ is stable. For $\delta_0$ is large enough, corresponding to Type III laser (far infrared), system (1.2) has a positive eigenvalue $-\frac{1}{2}(a+b) + \frac{1}{2}\sqrt{(a-b)^2 + 4\delta_0}$ at $E_0$, which implies that $E_0$ is unstable. When $\delta_0 = ab$, system (1.2) at $E_0$ has eigenvalues 0, $-c$ and $-(a+b)$. Notice that when $b = -a$, $\delta_0 < -a^2$ there is a pair of purely imaginary conjugate eigenvalues, which implies that $E_0$ is a nonhyperbolic equilibrium.

Note that system (1.2) is $\mathcal{C}_2$-symmetric with respect to the $x_3$-axis, i.e., invariant under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$. Thus, it suffices to discuss equilibrium $E_+$. Linearizing (1.2) at $E_+$, we get the characteristic polynomial

$$\Phi(\lambda) := \lambda^3 + (a+b+c)\lambda^2 + \frac{c(\delta_0 + a^2)}{a}\lambda + 2c(\delta_0 - ab).$$

According to the Routh–Hurwitz criterion [11], all eigenvalues have negative real parts if and only if

$$a + b + c > 0, \; \frac{c(\delta_0 + a^2)}{a} > 0, \; 2c(\delta_0 - ab) > 0, \; \frac{c(\delta_0 + a^2)(a+b+c)}{a} > 2c(\delta_0 - ab),$$

implying that $E_+$ is a locally stable node or focus in the case that $bc(\delta_0 + 3a^2) - \delta_0 c(a-c) + a^2 c(a+c) < 0$, $c(\delta_0 - ab) > 0$, $4c(\delta_0 + a^2) > 0$ and $a > 0$. Notice that

$$b = B(a, c, \delta_0) := \frac{\delta_0(a-c) - a^2(a+c)}{\delta_0 + 3a^2},$$

there is a pair of purely imaginary conjugate eigenvalues, which implies that $E_+$ is a non-hyperbolic equilibrium. When parameters lie in the set

$$\{(a, b, c, \delta_0) : b = B(a, c, \delta_0), a > 0, c(\delta_0 - ab) > 0, c(\delta_0 + a^2) > 0\}, \tag{2.1}$$

the equilibrium $E_+$ (i.e., $E_-$, in $\mathcal{C}_2$-symmetry to $E_+$ with respect to the $x_3$-axis) is of center-focus type because it has a pair of pure imaginary eigenvalues.

*2.1. Weak focus at $E_0$*

Consider parameters in the set

$$\mathcal{D}_0 := \{(a, b, c, \delta_0) : b = -a, \delta_0 < -a^2, c \neq 0\}. \tag{2.2}$$

Let $\varepsilon := b + a$ for $b$ near $-a$ and regard $\varepsilon$ as the perturbation parameter. In the case $\varepsilon = 0$ the equilibrium $E_0$ is a center-focus type and the linearization of system (1.2) at $E_0$ has eigenvalues $-c$ and $\pm\mathbf{i}\sqrt{-\delta_0 - a^2}$. Then, we can transform system (1.2) into the following form

$$\begin{cases} z_1' = -z_2 - a(\delta_0\sqrt{-\delta_0 - a^2})^{-1}z_1z_3 - \delta_0^{-1}z_2z_3, \\ z_2' = z_1 + a^2(\delta_0(-\delta_0 - a^2))^{-1}z_1z_3 + a(\delta_0\sqrt{-\delta_0 - a^2})^{-1}z_2z_3, \\ z_3' = -c(\sqrt{-\delta_0 - a^2})^{-1}z_3 + 4a(\delta_0\sqrt{-\delta_0 - a^2})^{-1}z_1^2 + 4\delta_0^{-1}z_1z_2, \end{cases} \tag{2.3}$$

where $z_i' := dz_i/d\tau_2$ and $d\tau_2 := \sqrt{-\delta_0 - a^2}d\tau_1$, by translating $E_0$ to the origin $O$ and diagonalizing the linear part of system (1.2) in the case that $\varepsilon = 0$.

Our first task is to approximate a center manifold. By Theorem 1 in [3], system (2.3) has a 2-dimensional center manifold

$$\mathcal{W}^c = \{(z_1, z_2, z_3)|z_3 = h(z_1, z_2),\ h(0,0) = 0,\ Dh(0,0) = 0\},$$

where $h : U \to \mathbb{R}$ is smooth on a neighborhood $U \subseteq \mathbb{R}^2$ of the origin as indicated in [3, p. 28]. Let $h(z_1, z_2) = \phi_2(z_1, z_2) + O(\|(z_1, z_2)\|^3)$, i.e., $\phi_2$ is the second order approximation of $h$. By [3, Theorem 3],

$$\begin{aligned}(M\phi_2)(z_1, z_2) :=& -\frac{\partial\phi_2}{\partial z_1}z_2 + \frac{\partial\phi_2}{\partial z_2}z_1 + \frac{c}{\sqrt{-\delta_0 - a^2}}\phi_2 - \frac{4a}{\delta_0\sqrt{-\delta_0 - a^2}}z_1^2 + \frac{4}{\delta_0}z_1z_2 \\ =& O(\|(z_1, z_2)\|^3).\end{aligned} \tag{2.4}$$

Comparing coefficients in (2.4), we get

$$\begin{aligned}\phi_2(z_1, z_2) = 4\{c\delta_0(4\delta_0 + 4a^2 - c^2)\}^{-1}\{&(2a\delta_0 - c\delta_0 + 2a^3 - a^2c - ac^2)z_1^2 \\ &- c(2a + c)\sqrt{-\delta_0 - a^2}z_1z_2 + (2a + c)(\delta_0 + a^2)z_2^2\}.\end{aligned} \tag{2.5}$$

Thus, by a substitution of (2.5), system (2.3) restricted to the manifold $\mathcal{W}^c$ can be presented as

$$\begin{cases} z_1' = -z_2 - 4\{c\delta_0^2(4\delta_0 + 4a^2 - c^2)\sqrt{-\delta_0 - a^2}\}^{-1}W(z_1, z_2), \\ z_2' = z_1 - 4a\{c\delta_0^2(4\delta_0 + 4a^2 - c^2)(\delta_0 + a^2)\}^{-1}W(z_1, z_2), \end{cases} \tag{2.6}$$

where

$$\begin{aligned}W(z_1, z_2) = a(2a\delta_0 - c\delta_0 + 2a^3 - a^2c - ac^2)z_1^3 + \sqrt{-\delta_0 - a^2}(&2a\delta_0 - c\delta_0 \\ + 2a^3 - 3a^2c - 2ac^2)z_1^2z_2 &+ (2a + c)(\delta_0 + a^2)(a + c)z_1z_2^2 \\ + \sqrt{-\delta_0 - a^2}(\delta_0 + a^2)&(2a + c)z_2^3 + o(\|(z_1, z_2)\|^3).\end{aligned}$$

Our second task is to compute a normal form for system (2.6). Let $z = z_1 + \mathbf{i}z_2$. Then (2.6) can be represented as the complex form

$$\dot{z} = \mathbf{i}z + 4(\sqrt{-\delta_0 - a^2} - a\mathbf{i})\{c\delta_0^2(4\delta_0 + 4a^2 - c^2)(\delta_0 + a^2)\}^{-1}W(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\mathbf{i}}). \tag{2.7}$$

Actually, there is no second degree terms. As done in [18], the third degree terms transformations with the near-identity transformation do not affect the coefficient of the third degree resonant term $z^2\bar{z}$. Hence, we can reduce system (2.7) to the form

$$\dot{w} = \mathbf{i}w + C_1 w^2 \bar{w} + O(\|(w, \bar{w})\|^5), \tag{2.8}$$

where

$$C_1 = \frac{\sqrt{-\delta_0 - a^2}\, c(2a - c) + \mathbf{i}(2c\delta_0 + 8a\delta_0 + 8a^3 + 2a^2 c - 3ac^2)}{2c\delta_0(4\delta_0 + 4a^2 - c^2)(-\delta_0 - a^2)}.$$

Thus, the first Lyapunov quantity $L_1$ is given by

$$L_1|_{E_0} = \mathrm{Re}(C_1) = \frac{2a - c}{2\delta_0(4\delta_0 + 4a^2 - c^2)\sqrt{-\delta_0 - a^2}}. \tag{2.9}$$

On the basis of the above results we can discuss the Hopf bifurcation for (1.2) at $E_0$. By the classical Hopf bifurcation theorem [5], we obtain

**Theorem 1.** *If $(a, b, c, \delta_0) \in \mathcal{D}_0$ and $a \neq c/2$ then the equilibrium $E_0$ of system (1.2) is a locally stable (respectively, unstable) weak focus of multiplicity 1 for $a < c/2$ (respectively, $a > c/2$). For sufficiently small $|\varepsilon|$ system (1.2) undergoes a Hopf bifurcation at $E_0$ as $\varepsilon$ passes 0. Moreover,*

**(1)** *for $a < c/2$, there is a unique stable limit cycle when $\varepsilon < 0$ but no limit cycle when $\varepsilon \geq 0$;*
**(2)** *for $a > c/2$, there is a unique unstable limit cycle when $\varepsilon > 0$ but no limit cycle when $\varepsilon \leq 0$.*

The Maxwell–Bloch system (1.2) looks like the general Lorenz system considered in [20] and it is really of the form in a special case as indicated in the Introduction, but it is quite different from the general Lorenz system. Equilibrium $E_0$, discussed in Theorem 1, can be of center-focus type but the corresponding one in [20] is not the case.

Consider the case $L_1 = 0$, i.e. $a = c/2$. Actually, the first three Lyapunov quantities $L_1$, $L_2$ and $L_3$ vanish in this case. We leave this case in Section 3.

### 2.2. Weak foci at $E_\pm$

Consider parameters in the set

$$\mathcal{D} := \{(a, b, c, \delta_0) : b = B(a, c, \delta_0), a > 0, c(\delta_0 - ab) > 0, c(\delta_0 + a^2) > 0\}. \tag{2.10}$$

Let $\epsilon := b - B(a, c, \delta_0)$ for $b$ near $(a, c, \delta_0)$ and regard $\epsilon$ as the perturbation parameter. When $\epsilon = 0$, i.e. $b = B(a, c, \delta_0)$, the linearization of system (1.2) at $E_+$ has eigenvalues $-2a(\delta_0 + a^2 + ac)/(\delta_0 + 3a^2)$ and $\pm\mathbf{i}\sqrt{c(\delta_0 + a^2)/a}$. For sufficiently small $|\epsilon|$, the linearization of (1.2) at $E_+$ has one negative eigenvalue and a pair of conjugate complex eigenvalues $\lambda_{1,2} = \sigma(\epsilon) \pm \mathbf{i}\,\omega(\epsilon)$ such that $\sigma(0) = 0$, $\omega(0) = \sqrt{c(\delta_0 + a^2)/a}$,

$$\frac{d\sigma}{d\epsilon}|_{\epsilon=0} = -\frac{c(\delta_0 + 3a^2)^3}{2\{4a^3(\delta_0 + a^2 + ac)^2 + [(\delta_0^2 + 4a^2\delta_0 + 3a^4)\sqrt{c/(\delta_0 + a^2)}]^2\}} \neq 0.$$

When $\epsilon = 0$, system (1.2) has a two-dimensional center manifold near $E_+$. Then, we can transform system (1.2) into the form

$$\begin{cases} z_1' = -z_2 + F_1(z_1, z_2, z_3), \\ z_2' = z_1 + F_2(z_1, z_2, z_3), \\ z_3' = -2a^2(\delta_0 + a^2 + ac)\{\sqrt{ac(\delta_0 + a^2)}(\delta_0 + 3a^2)\}^{-1} z_3 + F_3(z_1, z_2, z_3), \end{cases} \tag{2.11}$$

where $z_i' := dz_i/d\tau_2$, $d\tau_2 := \sqrt{c(\delta_0 + a^2)/a}\,d\tau_1$ and $F_i$ $(i = 1, 2, 3)$ are given in Appendix A, by translating $E_+$ to the origin $O$ and diagonalizing the linear part of system (1.2) in the case that $\epsilon = 0$.

Using the same procedure in Subsection 2.1 to compute the 6-th approximation of center manifold, we finally get the first three Lyapunov quantities

$$\begin{aligned} L_1|_{E_+} &= \frac{a^3 c(2a - c)(\delta_0 + 3a^2)^3 \Omega_1(a, c, \delta_0)}{8(c\delta_0 + 4a^3 + a^2 c)(\delta_0 + a^2 + ac)(\delta_0 + a^2)\sqrt{ac(\delta_0 + a^2)} K_1 K_2}, \\ L_2|_{E_+} &= \frac{a^4(2a - c)(\delta_0 + 3a^2)^5 \Omega_2(a, c, \delta_0)}{576(c\delta_0 + 4a^3 + a^2 c)^2(\delta_0 + a^2 + ac)(\delta_0 + a^2)^3\sqrt{ac(\delta_0 + a^2)} K_1^3 K_2^3 K_3}, \\ L_3|_{E_+} &= \frac{a^5(2a - c)(\delta_0 + 3a^2)^7 \Omega_3(a, c, \delta_0)}{331\,776c(c\delta_0 + 4a^3 + a^2 c)^6(\delta_0 + a^2 + ac)^5(\delta_0 + a^2)^5\sqrt{ac(\delta_0 + a^2)} K_1^5 K_2^5 K_3^2 K_4}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \Omega_1(a, c, \delta_0) &= \delta_0^4 + 5a(2a - c)\delta_0^3 + a^3(26a - 31c)\delta_0^2 + a^4(26a^2 - 71ac + 2c^2)\delta_0 \\ &\quad + 9a^7(a - 5c), \\ K_1 &= c\delta_0^3 + a^2(4a + 7c)\delta_0^2 + a^4(8a + 23c)\delta_0 + a^5(a + 4c)(4a + c), \\ K_2 &= c\delta_0^3 + a^2(a + 7c)\delta_0^2 + a^4(2a + 17c)\delta_0 + a^5(a^2 + 11ac + c^2), \\ K_3 &= 9c\delta_0^3 + a^2(4a + 63c)\delta_0^2 + a^4(8a + 143c)\delta_0 + a^5(4a^2 + 89ac + 4c^2), \\ K_4 &= 4c\delta_0^3 + a^2(a + 28c)\delta_0^2 + 2a^4(a + 31c)\delta_0 + a^5(a^2 + 38ac + c^2), \end{aligned}$$

and the polynomial $\Omega_2(a, c, \delta_0)$ of 232 terms, is given in Appendix A. The polynomial $\Omega_3(a, c, \delta_0)$ of 1026 terms, is too long to display, but interested readers can easily compute it using any available computer algebra system.

One can see a common factor $(2a - c)$ in $L_1$, $L_2$ and $L_3$. In this subsection we consider the case $a \neq \frac{c}{2}$. Since the denominators of $L_i$s $(i = 1, 2, 3)$ are both positive, real zeros of $L_i$ $(i = 1, 2, 3)$ are determined by $\Omega_i$s $(i = 1, 2, 3)$. Therefore, in order to determine the order of weak focus and give the corresponding parameter condition, we need to analyze the affine varieties $\mathbf{V}(\Omega_1, \Omega_2)$ and $\mathbf{V}(\Omega_1, \Omega_2, \Omega_3)$ (we recall that the variety of a polynomial ideal is the set of common zeros of polynomials generating the ideal).

**Theorem 2.** *If $(a, b, c, \delta_0) \in \mathcal{D}$ and $a \neq c/2$ the equilibrium $E_+$ of system (1.2) is a weak focus of order at most 3. Moreover, $E_+$ is of order $\ell$, $\ell = 1, 2, 3$, if and only if $(a, b, c, \delta_0) \in \mathcal{D}_\ell$, where*

$$\begin{aligned} \mathcal{D}_1 &:= \mathcal{D} \setminus \{(a, b, c, \delta_0) : a = \frac{c}{2}\} \setminus \mathcal{D}_2 \setminus \mathcal{D}_3, \\ \mathcal{D}_2 &:= \{(a, b, c, \delta_0) \in \mathcal{D} : \Omega_1 = 0, a \neq \zeta_i c, a \neq \frac{c}{2}, i = 1, 2\}, \\ \mathcal{D}_3 &:= \{(a, b, c, \delta_0) : b = B(a, c, \delta_0), \delta_0 = \Xi(a, c), a = \zeta_i c, c > 0, i = 1, 2\}, \end{aligned}$$

*$\Xi(a, c)$ is given in (2.13), $\zeta_1 \approx 7.100800902 \times 10^{-3}$ and $\zeta_2 \approx 3.781934043 \times 10^{-1}$.*

**Proof.** We first investigate when system (1.2) has a focus of order 3. Using the routine *minAssGTZ* (based on the algorithm of [12]) of the *primdec* library [8] of the computer algebra system SINGULAR [13] we find that the variety of the ideal $\langle \Omega_1, \Omega_2 \rangle$ consists of seven irreducible components defined by prime ideals $J_1, \ldots, J_7$ where $J_1 = \langle c - 2a, 3a^2 + \delta_0 \rangle$, $J_2 = \langle 4c^3 - 537c^2a + 156ca^2 - 1112a^3, 4c^2 - 529ca + 718a^2 + 162\delta_0 \rangle$, $J_3 = \langle 8c - 7a, 9a^2 + 2\delta_0 \rangle$, $J_4 = \langle c, 9a^4 + 8\delta_0 a^2 + \delta_0^2 \rangle$, $J_5 = \langle c, a^2 + \delta_0 \rangle$, $J_6 = \langle a, \delta_0 \rangle$, $J_7 = \langle g_1, \ldots, g_5 \rangle$ with

$$\begin{aligned} g_1 &= 4860c^7 - 742\,281c^6a + 8\,384\,728c^5a^2 - 33\,867\,084c^4a^3 + 57\,679\,952c^3a^4 \\ &\quad - 58\,329\,656c^2a^5 + 74\,779\,488ca^6 - 35\,189\,056a^7, \\ g_2 &= 22\,039\,674\,356\,198\,256\,539\,940c^6 - 3\,321\,283\,327\,197\,950\,417\,619\,759c^5a \\ &\quad + 31\,278\,840\,305\,921\,724\,111\,242\,728c^4a^2 - 92\,964\,622\,693\,414\,013\,509\,322\,164c^3a^3 \\ &\quad + 103\,113\,902\,470\,440\,177\,572\,551\,248c^2a^4 - 146\,701\,224\,872\,093\,995\,490\,671\,928ca^5 \\ &\quad + 139\,972\,292\,926\,740\,412\,342\,550\,192a^6 + 13\,545\,456\,502\,066\,787\,476\,381\,392\delta_0a^4, \\ g_3 &= 6\,851\,634\,739\,274\,520c^5 - 1\,036\,863\,732\,245\,476\,962c^4a + 10\,349\,861\,913\,622\,199\,318c^3a^2 \\ &\quad - 30\,877\,842\,609\,948\,852\,489c^2a^3 + 33\,376\,139\,523\,696\,940\,131ca^4 \\ &\quad - 42\,033\,748\,438\,313\,542\,876a^5 + 4\,534\,912\,418\,970\,834\,679\delta_0ca^2 \\ &\quad - 9\,237\,859\,684\,802\,411\,264\delta_0a^3, \\ g_4 &= -5\,071\,398\,660c^4 + 714\,534\,108\,871c^3a - 2\,843\,433\,053\,754c^2a^2 + 3\,850\,612\,802\,085ca^3 \\ &\quad - 10\,913\,918\,038\,772a^4 + 704\,900\,693\,341\delta_0c^2 - 988\,173\,957\,019\delta_0ca - 1\,566\,632\,018\,336\delta_0a^2, \\ g_5 &= -28\,136\,192\,400c^4 + 169\,379\,836\,258\,889c^2a^2 - 465\,492\,814\,856\,837ca^3 \\ &\quad - 990\,252\,677\,556\,100a^4 + 101\,652\,561\,937\,098\delta_0c^2 - 409\,287\,952\,913\,593\delta_0ca \\ &\quad + 138\,633\,004\,210\,400\delta_0a^2 + 32\,738\,653\,715\,544\delta_0^2. \end{aligned}$$

It is easily seen that the sets defined by ideals $J_4$, $J_5$, $J_6$ do not satisfy condition (2.10) and the set defined by $J_1$ does not satisfy the condition $a \neq \frac{c}{2}$ for the theorem. Straightforward computations show that for parameters from $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ we have $\Omega_3 \neq 0$.

The analysis of systems from $\mathbf{V}(J_7)$ is more difficult. In this case we first solve the equation $g_2 = 0$ for $\delta_0$, obtaining

$$\delta_0 = \Xi(a, c), \tag{2.13}$$

where

$$\begin{aligned} \Xi(a,c) = (&-139\,972\,292\,926\,740\,412\,342\,550\,192a^6 + 146\,701\,224\,872\,093\,995\,490\,671\,928a^5c \\ &- 103\,113\,902\,470\,440\,177\,572\,551\,248a^4c^2 + 92\,964\,622\,693\,414\,013\,509\,322\,164a^3c^3 \\ &- 31\,278\,840\,305\,921\,724\,111\,242\,728a^2c^4 + 3\,321\,283\,327\,197\,950\,417\,619\,759ac^5 \\ &- 22\,039\,674\,356\,198\,256\,539\,940c^6)/(13\,545\,456\,502\,066\,787\,476\,381\,392a^4). \end{aligned} \tag{2.14}$$

Then we substitute this value into $g_3$, $g_4$, $g_5$ and denoting the obtained functions by $\tilde{g}_i$ $(i = 3, 4, 5)$ we see that $\tilde{g}_i = f_i g_1$ where $f_i$ are rational functions. Thus, to find the solutions of the system $g_1 = \cdots = g_5 = 0$ it is sufficient to find solutions of the equation $g_1 = 0$. To this end, we denote $a = \zeta c$. Then $g_1(a, c) = c^7 \hat{g}_1(\zeta)$, where $\hat{g}_1(\zeta) := g_1(\zeta, 1)$. Using the MAPLE command "realroot($\hat{g}_1, 1/10^9$)" to isolate real zeros, we see that $\hat{g}_1$ has exactly three real zeros, which are covered by

$$I_1 := [\frac{3\,812\,213}{536\,870\,912}, \frac{7\,624\,427}{1\,073\,741\,824}],\ I_2 := [\frac{406\,082\,075}{1\,073\,741\,824}, \frac{101\,520\,519}{268\,435\,456}],\ I_3 := [\frac{1\,621\,125\,139}{1\,073\,741\,824}, \frac{405\,281\,285}{268\,435\,456}].$$

We claim that $\Theta := \Xi(\zeta, 1) + \zeta^2 \le 0$ in $I_3$ because $\mathcal{D}$ defined in (2.10), requires that $\Theta > 0$. Compute the derivative of $\Theta$,

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta} = -\frac{\Theta_1(\zeta)}{13\,545\,456\,502\,066\,787\,476\,381\,392\zeta^5}, \tag{2.15}$$

where

$$\begin{aligned}\Theta_1(\zeta) := (&252\,853\,672\,849\,347\,249\,732\,337\,600\zeta^6 - 146\,701\,224\,872\,093\,995\,490\,671\,928\zeta^5 \\ &+ 92\,964\,622\,693\,414\,013\,509\,322\,164\zeta^3 - 62\,557\,680\,611\,843\,448\,222\,485\,456\zeta^2 \\ &+ 9\,963\,849\,981\,593\,851\,252\,859\,277\zeta - 88\,158\,697\,424\,793\,026\,159\,760).\end{aligned}$$

In fact, using the Maple command "realroot($\Theta_1, 1/10^9$)", we find that there is no real zero in $I_3$, implying that $\Theta$ is monotone on $I_3$. Checking

$$\Theta\Big(\frac{1\,621\,125\,139}{1\,073\,741\,824}\Big) \approx -8.932757073 < 0, \quad \Theta\Big(\frac{405\,281\,285}{268\,435\,456}\Big) \approx -8.932757090 < 0,$$

we see that $\Theta(\zeta) < 0$ for $\zeta \in I_3$, the claim is true.

For $\zeta \in I_1 \bigcup I_2$, $\delta_0 = \Xi(a, c)$, $b = B(a, c, \delta_0)$ satisfies the requirement of $\mathcal{D} \setminus \Upsilon$. Therefore,

$$\mathbf{V}(J_7) = \{(a, b, c, \delta_0) : b = B(a, c, \delta_0), \delta_0 = \Xi(a, c), a = \zeta_i c, i = 1, 2\},$$

where $\zeta_1 \approx 7.100800902 \times 10^{-3} \in I_1$ and $\zeta_2 \approx 3.781934043 \times 10^{-1} \in I_2$. Thus, we see that

$$\begin{aligned}&V(\Omega_1, \Omega_2) \cap (\mathcal{D} \setminus \Upsilon) \\ &\quad = \{(a, b, c, \delta_0) : b = B(a, c, \delta_0), \delta_0 = \Xi(a, c), a = \zeta_i c, c > 0, i = 1, 2\}.\end{aligned} \tag{2.16}$$

Finally, we prove that $\mathbf{V}(\Omega_1, \Omega_2, \Omega_3) \cap (\mathcal{D} \setminus \Upsilon) = \emptyset$, that is, the maximal order of a week focus is 3. Computing with *minAssGTZ* of Singular the decomposition of the variety of the ideal $\langle \Omega_1, \Omega_2, \Omega_3 \rangle$ we find that it consists of 6 irreducible component defined by the ideals $J_1, \ldots, J_6$ given above. As we have seen above the zero sets of these ideals are outside the set $\mathcal{D} \setminus \Upsilon$, that is,

$$\mathbf{V}(\Omega_1, \Omega_2, \Omega_3) \cap (\mathcal{D} \setminus \Upsilon) = \emptyset. \tag{2.17}$$

Thus, $E_+$ is a weak focus of order at most 3. □

In the case of $a = c/2$, the first three Lyapunov quantities $L_1|_{E_+}$, $L_2|_{E_+}$ and $L_3|_{E_+}$ vanish. If we continue the same procedure as above, we may need to compute higher order Lyapunov quantities up to the first nonzero one. We meet the same problem as in Subsection 2.1. It is not easy to determine whether the equilibrium $E_0$ is a center or focus by computing approximations of center manifold. We consider this case in Section 3.

## 3. Centers

Determining whether an equilibrium with a pair of conjugate pure imaginary eigenvalues is really a center is much more difficult in a higher dimensional space than on a plane. As we know, one hardly identifies centers by proving all Lyapunov quantities to be zeros, not mentioning the determination on an approximated center manifold. Although time-reversibility or integrability are applied to judge centers for planar systems, they are not available on an approximated center manifold. For this reason, we need to give a closed form for a global center manifold.

The above mentioned global center manifold is selected from invariant algebraic surfaces. It is easy to check (see, e.g. [24, Proposition 3.6.2]) that a surface defined by the equation $F(x_1, x_2, x_3) = 0$, where $F$ is a polynomial, is an invariant surface of the system

$$\begin{cases} \dot{x}_1 = P(x_1, x_2, x_3), \\ \dot{x}_2 = R(x_1, x_2, x_3), \\ \dot{x}_3 = Q(x_1, x_2, x_3), \end{cases} \tag{3.1}$$

where the maximal degree of polynomials $P$, $Q$, $R$ is $m$, if and only if

$$\frac{\partial F}{\partial x_1} P + \frac{\partial F}{\partial x_2} Q + \frac{\partial F}{\partial x_3} R = K\,F, \tag{3.2}$$

with $K$ being a polynomial of degree at most $m - 1$. It is clear that $F$ is a partial integral of (3.1). The polynomial $F$ is called a Darboux polynomial [7] of system (3.1) and $K$ is termed a cofactor.

Below we find algebraic partial integrals up to degree 2 of Maxwell–Bloch system (1.2) passing through $E_0$. By translating equilibrium $E_0$ to the origin, system (1.2) is changed into the following system

$$\begin{cases} \dot{x}_1 = -a\,x_1 + x_2 & := P(x_1, x_2, x_3), \\ \dot{x}_2 = \delta_0\,x_1 - b\,x_2 + x_1 x_3 & := R(x_1, x_2, x_3), \\ \dot{x}_3 = -c\,x_3 - 4\,x_1 x_2 & := Q(x_1, x_2, x_3). \end{cases} \tag{3.3}$$

The following theorem gives the conditions on coefficients of system (1.2) which yield the existence of invariant algebraic surfaces. As it turns out, some of the obtained surfaces are also center manifolds of the system.

**Theorem 3.** *The Maxwell–Bloch system (1.2) has an invariant algebraic curve or surface $F = 0$ passing through the point $E_0$ and defined by a polynomial of degree at most 2 if and only if one of the following conditions holds:*

**(1)** $b = c$ *and* $\delta_0 = 0$, *under which* $F(x_1, x_2, x_3) = 4x_2^2 + x_3^2$, *or*
**(2)** $a = b = c$, *under which* $F(x_1, x_2, x_3) = 4\delta_0 x_1^2 - 4x_2^2 - (x_3 - \delta_0)^2$, *or*
**(3)** $a = c/2$, *under which* $F(x_1, x_2, x_3) = 2x_1^2 + x_3 - \delta_0$, *or*
**(4)** $b = c = 0$, *under which* $F(x_1, x_2, x_3) = 4x_2^2 + x_3^2 - \nu$, *where* $\nu \geq 0$ *is an arbitrary constant.*

**Proof.** Our strategy is to find a Darboux factor $F$ of system (3.3) in the polynomial form

$$F(x_1, x_2, x_3) = \sum_{i=1}^{l} F_i(x_1, x_2, x_3),$$

where $l \geq 1$ is an integer, $F_i$ is a homogeneous polynomial of degree $i$, and $F_l \neq 0$. By the $\mathcal{C}_2$-symmetry of system (3.3) with respect to the $x_3$-axis, $F(-x_1, -x_2, x_3) = F(x_1, x_2, x_3)$, which implies that $F$ is of the form

$$F(x_1, x_2, x_3) = \sum_{k+m+n=1}^{l} a_{k,m,n} x_1^k x_2^m x_3^n, \quad k + m \equiv 0 \pmod 2. \tag{3.4}$$

On the other hand, as mentioned just before the theorem, the cofactor $K(x_1, x_2, x_3)$ is of at most 1 order and therefore we assume that

$$K(x_1, x_2, x_3) = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3. \tag{3.5}$$

In order to prove the existence of such an $F(x_1, x_2, x_3)$, we start our searching from $l = 1$ upon. For $l = 1$, $F(x_1, x_2, x_3) = a_{0,0,1} x_3$ by (3.4). Substituting this $F$ and the cofactor $K$ given in (3.5) in Eq. (3.2) and comparing the coefficients of the term $x_1 x_2$, we obtain $a_{0,0,1} = 0$, implying that $F(x_1, x_2, x_3) \equiv 0$, a contradiction to the assumption that $F_l(x_1, x_2, x_3) \neq 0$. Next, consider $l = 2$. Then,

$$F(x_1, x_2, x_3) = a_{0,0,1} x_3 + a_{2,0,0} x_1^2 + a_{0,2,0} x_2^2 + a_{0,0,2} x_3^2 + a_{1,1,0} x_1 x_2.$$

Substituting in (3.2) this $F(x_1, x_2, x_3)$ and the $K(x_1, x_2, x_3)$ given in (3.5) and comparing the coefficients of all terms, we obtain the algebraic system

$$g_j(a, b, c, \delta_0, a_{s_1,s_2,s_3}, c_m) = 0, \quad j = 1, \ldots, 17, \tag{3.6}$$

where

$$\begin{aligned}
&g_1 = -a_{0,0,1}(c + c_0), \quad g_2 = -(2a + c_0)a_{2,0,0} + \delta_0 a_{1,1,0}, \quad g_3 = -c_1 a_{0,0,1},\\
&g_4 = -4a_{0,0,1} + 2a_{2,0,0} - (a + b + c_0)a_{1,1,0} + 2\delta_0 a_{0,2,0}, \quad g_5 = a_{1,1,0} - (2b - c_0)a_{0,2,0},\\
&g_6 = -c_2 a_{0,0,1}, \quad g_7 = -c_3 a_{0,0,1} - (2c - c_0)a_{0,0,2}, \quad g_8 = -c_1 a_{2,0,0},\\
&g_9 = -c_2 a_{2,0,0} - c_1 a_{1,1,0}, \quad g_{10} = -c_3 a_{2,0,0} + a_{1,1,0}, \quad g_{11} = -c_2 a_{1,1,0} - c_1 a_{0,2,0},\\
&g_{12} = -c_1 a_{0,0,2}, \quad g_{13} = -c_3 a_{1,1,0} + 2a_{0,2,0} - 8a_{0,0,2}, \quad g_{14} = -c_2 a_{0,2,0},\\
&g_{15} = -c_3 a_{0,2,0}, \quad g_{16} = -c_2 a_{0,0,2}, \quad g_{17} = -c_3 a_{0,0,2}.
\end{aligned} \tag{3.7}$$

We denote by $I := \langle g_1, g_2, \ldots, g_{17} \rangle$ the ideal generated by polynomials given above. To obtain the condition for existence of Darboux polynomial (3.4) we have to eliminate from the system

$$g_1 = g_2 = \cdots = g_{17} \tag{3.8}$$

the variables $a_{ijk}$ and $c_m$. One way is to use resultants. However doing this one has to deal with very cumbersome and unfeasible calculations. Much more efficient way is provided by the so-called Elimination Theorem (see, e.g. [6,24]). By the theorem in order to eliminate from the system the $a_{ijk}$ and $c_m$ it is sufficient to compute a Gröbner basis of the ideal with respect to an elimination order where $a_{ijk}, c_m > a, b, c, \delta_0$. The algorithm is realized in many computer algebra system, for our computations we used the routine *eliminate* of the computer algebra system SINGULAR [13].

However if one tries to use the algorithm directly, the output of computation is the zero ideal $\langle 0 \rangle$, that means, the system always has a solution. Indeed, system (3.7) always has the trivial solution, so, in order to get conditions for existence of invariant surfaces we have to exclude this solution from the consideration. To this end we have to look for solutions of (3.8) under which of conditions: $a_{2,0,0} \neq 0$, $a_{0,2,0} \neq 0$, $a_{0,0,2} \neq 0$, $a_{1,1,0} \neq 0$. Thus, we first add to the ideal $I$ the polynomial $1 - w a_{2,0,0}$, obtaining the ideal $I_1 = \langle 1 - w a_{2,0,0}, I \rangle$ in the polynomial ring over the field of rational numbers. Eliminating from $I_1$ by *eliminate* of SINGULAR the variables $a_{r_1,r_2,r_3}$, $c_m$ and $w$ we obtain the elimination ideal $J_1 = \langle f_1, f_2 \rangle$, where

$$\begin{aligned}
f_1 &:= 8ab^2 - 10abc + 2ac^2 - 4b^2c + 5bc^2 - c^3,\\
f_2 &:= 6a^2 - 4ab - 5ac + 2bc + c^2.
\end{aligned}$$

In a similar way, after elimination of $a_{r_1,r_2,r_3}$, $c_m$ and $w$ from the ideal $I_2 = \langle 1 - wa_{0,2,0}, I\rangle$ we obtain the ideal $J_2 = \langle b - c, a\,c\,d - c^2 d\rangle$. With the routine *intersect* of SINGULAR we then compute $J = J_1 \cap J_2$ (thus, $V(J) = V(J_1) \cup V(J_2)$). Now, using the routine *minAssGTZ* of SINGULAR to find the decomposition of the affine variety $V(J)$ of the ideal $J$ we obtain

$$V(J) = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4, \tag{3.9}$$

where

$$V_1 = \{(a, b, c, \delta_0) : b = c, \delta_0 = 0\}, \quad V_2 = \{(a, b, c, \delta_0) : a = b = c\},$$
$$V_3 = \{(a, b, c, \delta_0) : a = \frac{c}{2}\}, \quad V_4 = \{(a, b, c, \delta_0) : b = 0, c = 0\}.$$

Computing for the remaining cases $a_{0,0,2} \neq 0$ and $a_{1,1,0} \neq 0$ we do not obtain conditions different from those defined by (3.9).

Performing now easy computations we find the Darboux polynomial given in the statement in the theorem, corresponding to parameters of the sets $V_1$, $V_2$, $V_3$, $V_4$. Note, that despite of the polynomial of case **(1)** has degree 2, the equation $F = 0$ defines an invariant line, but not an invariant surface. □

In what follows we discuss system (1.2) in the four cases listed in Theorem 3.

### *3.1. Case (1) and generic case (2)*

System (1.2) restricted to the invariant line $4x_2^2 + x_3^2 = 0$ (equivalently, $x_2 = x_3 = 0$), given in case **(1)** of Theorem 3, is the equation $\dot{x}_1 = -a\,x_1$, which has no centers obviously.

Another invariant surface $4\delta_0 x_1^2 - 4x_2^2 - (x_3 - \delta_0)^2 = 0$, given in case **(2)** of Theorem 3, has two possibilities: for $\delta_0 < 0$ it is exactly the equilibrium $E_0 : (0, 0, \delta_0)$, which is not a center as indicated in the second paragraph of Section 2; for $\delta_0 > 0$, it has two branches

$$\mathcal{F}_+ : \ x_1 = \{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2}, \quad \mathcal{F}_- : \ x_1 = -\{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2},$$

restricted to which system (1.2) can be presented as

$$\begin{cases} \dot{x}_2 = -c\,x_2 + x_3\{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2}, \\ \dot{x}_3 = -c\,(x_3 - \delta_0) - 4\,x_2\{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \dot{x}_2 = -c\,x_2 - x_3\{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2}, \\ \dot{x}_3 = -c\,(x_3 - \delta_0) + 4\,x_2\{\frac{1}{\delta_0}x_2^2 + \frac{1}{4\delta_0}(x_3 - \delta_0)^2\}^{1/2}, \end{cases} \tag{3.11}$$

respectively. The generic case is that $c \neq 0$, where system (3.10) has exactly one equilibrium $S_1 : (0, \delta_0)$ when $\delta_0 \leq c^2$ but $S_1$ and one more equilibrium $S_2 : (c\sqrt{\delta_0 - c^2}/2, c^2)$ when $\delta_0 > c^2$. Neither $S_1$ nor $S_2$ is a center because the Jacobian matrix has the traces $-2c$ and $-c$ at $S_1$ and $S_2$ respectively. Similarly, system (3.11) has exactly one equilibrium $S_1 : (0, \delta_0)$ when $\delta_0 \leq c^2$ but $S_1$ and one more equilibrium $S_3 : (-c\sqrt{\delta_0 - c^2}/2, c^2)$ when $\delta_0 > c^2$. Again, neither $S_1$ nor $S_3$ is a center because the Jacobian matrix has the traces $-2c$ and $-c$ at $S_1$ and $S_3$ respectively. The special case $c = 0$ will be left to Subsection 3.4.

### 3.2. Generic case (3)

In this case we have $a = c/2$, where system (1.2) has an invariant surface $\mathcal{F}_1 : \Psi_1(x_1, x_2, x_3) := 2x_1^2 + x_3 - \delta_0 = 0$ by Theorem 3. As above, we first consider the generic situation that $c \neq 0$.

**Theorem 4.** *In the case $a = c/2$ and $(a, b, c, \delta_0) \in \mathcal{D}_0 \cup \mathcal{D}$, where $\mathcal{D}_0$ and $\mathcal{D}$ are defined in (2.2) and (2.10), the invariant surface $\mathcal{F}_1$ is a global center manifold of system (1.2) and, restricted to the manifold, the equilibrium $E_0$ is a center of system (1.2) if $(a, b, c, \delta_0) \in \mathcal{D}_0$ and the equilibria $E_\pm$ are both centers of system (1.2) if $(a, b, c, \delta_0) \in \mathcal{D}$. Moreover,*

**(1)** *$E_0$ is a rough center, i.e., the order of center is 0, if $\delta_0 \neq -c^2(c^2 + 1)/4$,*
**(2)** *$E_0$ is a weak center of order 1 if $\delta_0 = -c^2(c^2 + 1)/4$ and $c^2 \neq 4\sqrt{15}/15 - 1$,*
**(3)** *$E_0$ is a weak center of order 2 if $\delta_0 = -c^2(c^2 + 1)/4$ and $c^2 = 4\sqrt{15}/15 - 1$, and*
**(4)** *$E_\pm$ are both rough centers.*

**Proof.** First, we prove the surface $\mathcal{F}_1$ is a global center manifold of system (1.2). Actually, when $a = c/2$ and $(a, b, c, \delta_0) \in \mathcal{D}_0$, the equilibrium $E_0$ is of center-focus type. Computing the gradient of the function $\Psi_1(x_1, x_2, x_3)$ at $E_0$, we obtain $\nabla\Psi_1(E_0) = (0, 0, 1)$, which is a normal vector of the surface $\mathcal{F}_1$ at $E_0$. On the other hand, the tangent space of center manifolds [3] at $E_0$ is spanned by the vectors

$$\mathbf{e}_1 = (-\frac{c}{2\delta_0}, 1, 0), \quad \mathbf{e}_2 = (-\frac{\sqrt{-4\delta_0 - c^2}}{4\delta_0}, 0, 0),$$

the eigenvectors of the linear part of system (1.2) at $E_0$ corresponding to the pair of conjugate pure imaginary eigenvalues. One can check that

$$\nabla\Psi_1(E_0) \cdot \mathbf{e}_1 = 0, \quad \nabla\Psi_1(E_0) \cdot \mathbf{e}_2 = 0,$$

implying that the surface $\mathcal{F}_1$ also has the same tangent space at $E_0$. Therefore, the surface $\mathcal{F}_1$ is a global center manifold of system (3.12). Similarly, when $a = c/2$ and $(a, b, c, \delta_0) \in \mathcal{D}$, we can check that $E_0$ and $E_\pm$ all lie on the surface $\mathcal{F}_1$ and the equilibria $E_\pm$ are both of center-focus type. Simple computation shows that the normal vector of the surface $\mathcal{F}_1$ at $E_+$ is $\nabla\Psi_1(E_+) = (\sqrt{8\delta_0 + 2c^2}, 0, 1)$ and that the tangent space of center manifolds [3] at $E_+$ is spanned by the vectors

$$\mathbf{e}_1 = (0, -\frac{1}{2}, 0), \quad \mathbf{e}_2 = (-\frac{1}{\sqrt{8\delta_0 + 2c^2}}, -\frac{c}{2\sqrt{8\delta_0 + 2c^2}}, 1).$$

One can check that

$$\nabla\Psi_1(E_+) \cdot \mathbf{e}_1 = 0, \quad \nabla\Psi_1(E_+) \cdot \mathbf{e}_2 = 0,$$

which implies the surface $\mathcal{F}_1$ also has the same tangent space at $E_+$. Thus, the surface $\mathcal{F}_1$ is a global center manifold of system (1.2) in this situation.

Next, we prove that $E_0$ is a center on the center manifold $\mathcal{F}_1$ if $a = c/2$ and $(a, b, c, \delta_0) \in \mathcal{D}_0$ and that $E_\pm$ are both centers on the center manifold $\mathcal{F}_1$ if $a = c/2$ and $(a, b, c, \delta_0) \in \mathcal{D}$. Restricted to the manifold $\mathcal{F}_1$, system (1.2) becomes

$$\begin{cases} \dot{x}_1 = -\frac{c}{2}x_1 + x_2, \\ \dot{x}_2 = \delta_0 x_1 - b\, x_2 - 2x_1^3. \end{cases} \tag{3.12}$$

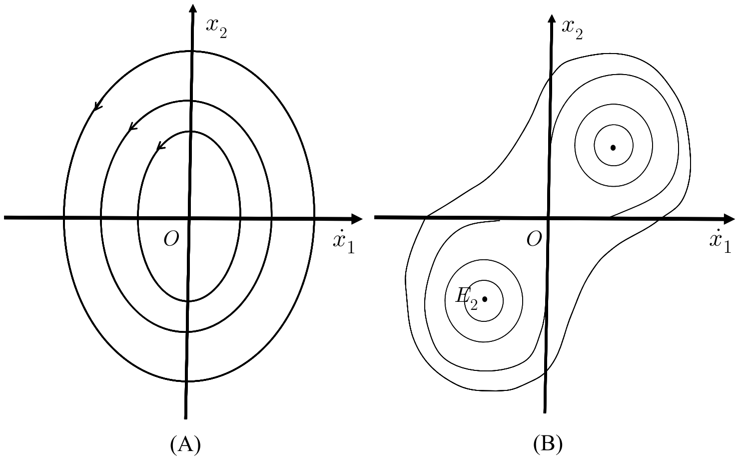


**Fig. 1.** (A) Qualitative properties of $E_0$; (B) Qualitative properties of $E_0$ and $E_\pm$ for system (1.2) on center manifold $\mathcal{F}_1$.

If $(a, b, c, \delta_0) \in \mathcal{D}_0$, system (3.12) has exactly one equilibrium $O : (0, 0)$, which corresponds to $E_0$ of (1.2). System (3.12) can be changed by an invertible linear transformation into the following form

$$\begin{cases} x_1' = -x_2 + \{2\delta_0^3\sqrt{-4\delta_0 - c^2}\}^{-1}(c\,x_1 + \sqrt{-4\delta_0 - c^2}x_2)^3, \\ x_2' = x_1 + c\{2\delta_0^3(4\delta_0 + c^2)\}^{-1}(c\,x_1 + \sqrt{-4\delta_0 - c^2}x_2)^3, \end{cases} \tag{3.13}$$

with the canonical linear part, where $x_i' := dx_i/d\tau_2$, $d\tau_2 := \sqrt{-\delta_0 - c^2/4}d\tau_1$, and $a = -b = c/2$. With the change of variables

$$x = \frac{c}{2}x_1 + \sqrt{-\delta_0 - \frac{1}{4}c^2}x_2,\ y = \sqrt{-\delta_0 - \frac{1}{4}c^2}x_1 - \frac{c}{2}x_2,$$

system (3.13) is further changed into the form

$$\begin{cases} x' = y, \\ y' = -x + 8\{\delta_0^2(4\delta_0 + c^2)\}^{-1}x^3, \end{cases} \tag{3.14}$$

which has the Hamiltonian $H(x, y) = y^2/2 + U(x)$, where $U(x) = x^2/2 - 2x^4/\delta_0^2(4\delta_1 + c^2)$. Note that $U'(0) = 0$ and $U''(0) = 1 > 0$, implying that the potential $U$ reaches a strictly local minimum at the equilibrium $O$. It follows that $O$ is a center of system (3.14), i.e., $E_0$ is a center of system (1.2) restricted to the center manifold $\mathcal{F}_1$ (see in Fig. 1 (A)).

If $(a, b, c, \delta_0) \in \mathcal{D}$, system (3.12) has three equilibria $O$ and

$$S_+ : (\frac{\sqrt{2\delta_0 - bc}}{2}, \frac{c\sqrt{2\delta_0 - bc}}{4}), \quad S_- : (-\frac{\sqrt{2\delta_0 - bc}}{2}, -\frac{c\sqrt{2\delta_0 - bc}}{4}),$$

which correspond to $E_0$ and $E_\pm$ of (1.2), respectively. Translating $S_+$ to the origin $O$ and diagonalizing the linear part, we can change system (3.12) into the form

$$\begin{cases} x_1' = -x_2 - \frac{12(c\,x_1 + \sqrt{8\delta_0 + 2c^2}x_2)^2}{(8\delta_0 + 3c^2)^2} - \frac{32(c\,x_1 + \sqrt{8\delta_0 + 2c^2}x_2)^3}{(8\delta_0 + 3c^2)^3\sqrt{8\delta_0 + 2c^2}}, \\ x_2' = x_1 + \frac{12(c\,x_1 + \sqrt{8\delta_0 + 2c^2}x_2)^2}{(8\delta_0 + 3c^2)^2\sqrt{8\delta_0 + 2c^2}} + \frac{16\,c(c\,x_1 + \sqrt{8\delta_0 + 2c^2}x_2)^3}{(8\delta_0 + 3c^2)^3(4\delta_0 + c^2)}, \end{cases} \tag{3.15}$$

where $x_i' := dx_i/d\tau_2$, $d\tau_2 := \sqrt{2\delta_0 + c^2/2}\,d\tau_1$ and the equalities $a = -b = c/2$ given in the region $\mathcal{D}$ is used. With the change of variables

$$x = c\,x_1 + \sqrt{8\delta_0 + 2c^2}x_2,\ \ y = \sqrt{8\delta_0 + 2c^2}\,x_1 - c\,x_2,$$

system (3.15) is further changed into the form

$$\begin{cases} x' = y, \\ y' = -x - \frac{12}{(8\delta_0+3c^2)\sqrt{8\delta_0+2c^2}}x_1^2 - \frac{16}{(8\delta_0+3c^2)^2(4\delta_0+c^2)}x_1^3, \end{cases} \tag{3.16}$$

which has the Hamiltonian $H(x, y) = y^2/2 + U(x)$, where

$$U(x) = \frac{1}{2}x^2 + \frac{4}{(8\delta_0 + 3c^2)\sqrt{8\delta_0 + 2c^2}}x^3 + \frac{4}{(8\delta_0 + 3c^2)^2(4\delta_0 + c^2)}x^4.$$

Note that $U'(0) = 0$ and $U''(0) = 1 > 0$, implying that the potential $U$ reaches a strictly local minimum at the equilibrium $O$. It follows that $O$ is a center of system (3.16), i.e., $E_+$ is a center of system (1.2) restricted to the center manifold $\mathcal{F}_1$ (see in Fig. 1 (B)). Similarly, $E_-$ is also a center on the center manifold.

In order to determine the order of center $E_0$, let $P(r, \mu)$ denote the minimum period of the periodic orbit passing through the nonzero point $(r, 0)$, which surrounds the center $O$ of (3.13), where $\mu := (a, b, c, \delta_0)$. Note that the time-rescaling between (3.12) and (3.13) does not change the monotonicity of the period function $P(r, \mu)$ except for a constant multiplier $\sqrt{-\delta_0 - c^2/4}$. By [4, Lemma 2.1],

$$P(r, \mu) = 2\pi + \sum_{i=2}^{\infty} \iota_i(\mu)r^i, \tag{3.17}$$

where $\iota_i(\mu)$'s are called *period coefficients* or *period quantities*. For $\mu \in \mathcal{D}_0$, using the polar coordinates $z_1 = r\cos\vartheta$ and $z_2 = r\sin\vartheta$ in (3.13), we get $r' = G_3(\vartheta)r^3$, $\vartheta' = 1 + H_2(\vartheta)r^2$, where

$$\begin{aligned} G_3(\vartheta) :=\ & \frac{1}{2\delta_0^3\sqrt{-4\delta_0 - c^2}}\big\{ - 4c(-3\delta_0 - c^2 + (\delta_0 + c^2)\sqrt{-4\delta_0 - c^2})\cos^4\vartheta \\ & + 4(c^2(3\delta_0 + c^2) + (\delta_0 + c^2)\sqrt{-4\delta_0 - c^2})\cos^3\vartheta\sin\vartheta \\ & + c(-3(4\delta_0 + c^2) + (8\delta_0 + 5c^2)\sqrt{-4\delta_0 - c^2})\cos^2\vartheta \\ & - (4\delta_0 + c^2)(3c^2 + \sqrt{-4\delta_0 - c^2})\cos\vartheta\sin\vartheta - c(4\delta_0 + c^2)\sqrt{-4\delta_0 - c^2}\big\}, \\ H_2(\vartheta) :=\ & \frac{1}{2\delta_0^3\sqrt{-4\delta_0 - c^2}}\big\{4(c^2(3\delta_0 + c^2) + (\delta_0 + c^2)\sqrt{-4\delta_0 - c^2})\cos^4\vartheta \\ & + 4c(-(3\delta_0 + c^2) + (\delta_0 + c^2)\sqrt{-4\delta_0 - c^2})\cos^3\vartheta\sin\vartheta \\ & - (3c^2(4\delta_0 + c^2) + (8\delta_0 + 5c^2)\sqrt{-4\delta_0 - c^2})\cos^2\vartheta \\ & - (4\delta_0 + c^2)(-3 + \sqrt{-4\delta_0 - c^2})\cos\vartheta\sin\vartheta + (4\delta_0 + c^2)\sqrt{-4\delta_0 - c^2}\big\}. \end{aligned}$$

Then, using $G_3$ and $H_2$, one can compute the period quantities

$$\begin{aligned} \iota_2(\mu) =\ & -\frac{3\pi}{2}\frac{\sqrt{-4\delta_0 - c^2} - c^2}{\delta_0^2\sqrt{-4\delta_0 - c^2}}, \\ \iota_4(\mu) =\ & \frac{3\pi}{8}\delta_0^{-5}(4\delta_0 + c^2)^{-1}\big\{(15c^2 + 35 + 24c\pi)\delta_0^2 + c^2(-5c^2 + 12c\pi + 5)\delta_0 \\ & - c^4(c^2 + 1) + 2c\delta_0(3\pi(c^2 - 1) + 5c)\sqrt{-4\delta_0 - c^2}\big\}, \\ \iota_6(\mu) =\ & \frac{3\pi}{128}\delta_0^{-8}(4\delta_0 + c^2)^{-3/2}\big\{[(1120c^3 + 768\pi^2c^3 + 2160\pi c^2 - 1536\pi^2c + 2380c \end{aligned}$$

$$
\begin{aligned}
&- 2160\pi)\delta_0^3 + 3c^2(64\pi^2c^3 - 35c^3 + 360\pi c^2 - 192\pi^2 c + 105c - 120\pi)\delta_0^2 \\
&+ 2c^4(-13c^3 + 24\pi c^2 - 35c + 24\pi)\delta_0 + c^7(11c^2 + 3)]c + [(-720\pi c^3 \\
&- 768\pi^2 c^2 - 1120c^2 c - 2160\pi - 1540)\delta_0^3 - 3c^2(-120\pi c^3 + 192\pi^2 c^2 \\
&- 105c^2 + 360\pi c - 64\pi^2 + 35)\delta_0^2 + 2c^4(24\pi c^3 + 25c^2 + 24\pi c + 35)\delta_0 \\
&- c^6(13c^2 + 5)]\sqrt{-4\delta_0 - c^2}\}.
\end{aligned}
$$

Thus, as defined in [4], $E_0$ is a center of order 0 if and only if $\iota_2(\mu) \neq 0$, i.e., $\delta_0 \neq -c^2(c^2+1)/4$. In case $\delta_0 = -c^2(c^2+1)/4$, $\iota_2(\mu) = 0$ and

$$
\iota_4(\mu) = \frac{24\pi(15c^4 + 30c^2 - 1)}{c^{10}(c^2+1)^4},
$$

implying that $E_0$ is a center of order 1 if and only if $\delta_0 = -c^2(c^2+1)/4$ and $c^2 \neq 4\sqrt{15}/15 - 1$. Further, if $\delta_0 = -c^2(c^2+1)/4$ and $c^2 = 4\sqrt{15}/15 - 1$, we have $\iota_2(\mu) = \iota_4(\mu) = 0$ and

$$
\iota_6(\mu) = \frac{345\,990\,234\,375\sqrt{15}(2\sqrt{15} - 5)\pi}{8(4\sqrt{15} - 15)} \neq 0,
$$

implying that $E_0$ is a center of order 2.

We similarly discuss for $\mu \in \mathcal{D}$. One can compute

$$
\iota_2(\mu) = \frac{48\pi}{(4\delta_0 + c^2)(8\delta_0 + 3c^2)} \neq 0,
$$

implying that the center $E_+$ is of order 0, i.e., a rough center. This completes the proof. □

### *3.3. Generic case (4)*

In this case we have $b = c = 0$, where system (1.2) has the invariant surface $\mathcal{F}_2 : \Psi_2(x_1, x_2, x_3) := 4x_2^2 + x_3^2 = \nu$ ($\forall \nu \geq 0$), i.e., $4x_2^2 + x_3^2$ is a first integral, by Theorem 3. As above, we generically assume that $a \neq 0$ and leave the special case $a = b = c = 0$ to Subsection 3.4. The surface $\mathcal{F}_2$ has two possibilities: for $\nu = 0$ it is the straight line $x_2 = x_3 = 0$, i.e., the $x_1$-axis; for $\nu > 0$, it has two branches $\mathcal{F}_{2+} :\ x_3 = \{\nu - 4x_2^2\}^{1/2}$ and $\mathcal{F}_{2-} :\ x_3 = -\{\nu - 4x_2^2\}^{1/2}$. System (1.2) restricted to $\mathcal{F}_{2+}$ and $\mathcal{F}_{2-}$ can be presented as

$$
\begin{cases} \dot{x}_1 = -a\,x_1 + x_2, \\ \dot{x}_2 = x_1\{\nu - 4x_2^2\}^{1/2} \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_1 = -a\,x_1 + x_2, \\ \dot{x}_2 = -x_1\{\nu - 4x_2^2\}^{1/2}, \end{cases} \tag{3.18}
$$

respectively. A similar discussion to (3.10) and (3.11) shows that neither of systems in (3.18) has a center. In fact, the first system of (3.18) has three equilibria $S_{11} : (0,0)$, $S_{12} : (\sqrt{\nu}/2a, \sqrt{\nu}/2)$ and $S_{13} : (-\sqrt{\nu}/2a, -\sqrt{\nu}/2)$ but the latter two are not equilibria of system (1.2). Moreover, $S_{11}$ is not a center because the Jacobian matrix has the trace $-a$. The second system of (3.18) can be discussed similarly.

For $\nu = 0$, the invariant surface $\mathcal{F}_2$ is the $x_1$-axis, but we cannot restrict our system to a one-dimensional invariant manifold to judge whether an equilibrium on it is a center. We turn to consider the singular line $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0, x_2 = 0\}$, which was given in the beginning of Section 2. Clearly, the $x_1$-axis intersects the singular line at equilibrium $O : (0, 0, 0)$. A simple computation shows that the eigenvalues at $O$ are 0, 0 and $-a$. Diagonalizing the linear part of system (1.2), we obtain

$$\begin{cases} x_1' = 4a^{-2}x_2^2 + 4a^{-1}x_2x_3, \\ x_2' = -a^{-2}x_1x_2 - a^{-1}x_1x_3, \\ x_3' = x_3 + a^{-3}x_1x_2 + a^{-2}x_1x_3, \end{cases} \tag{3.19}$$

where $d\tau_2 = -ad\tau_1$. By Theorem 1 in [3], system (3.19) has a 2-dimensional center manifold

$$\mathcal{W}^c = \{(x_1, x_2, x_3)|x_3 = h(x_1, x_2),\ h(0, 0) = 0,\ Dh(0, 0) = 0\}, \tag{3.20}$$

where $h : U \to \mathbb{R}$, defined on a small neighborhood $U \subseteq \mathbb{R}^2$ of the origin, is of enough smoothness. Let $h(x_1, x_2) = \phi_k(x_1, x_2) + O(\|(x_1, x_2)\|^{k+1})$ and $\phi_k(x_1, x_2) = \Sigma_{i+j\le k}\gamma_{i,j}x_1^ix_2^j$. By Theorem 3 in [3], $(M\phi_k)(x_1, x_2) = O(\|(x_1, x_2)\|^{k+1})$, where the operator $M$ is defined by

$$\begin{aligned}(Mh)(x_1, x_2) := \frac{\partial h}{\partial x_1}(4a^{-2}x_2^2 + 4a^{-1}x_2h) + \frac{\partial h}{\partial x_2}(-a^{-2}x_1x_2 - a^{-1}x_1h) \\ - h - a^{-3}x_1x_2 + a^{-2}x_1h.\end{aligned}$$

**Lemma 1.** *Function $h(u, v)$ defined in (3.20) satisfies $h(u, -v) = -h(u, v)$.*

**Proof.** First, we claim that $h(u, 0) = 0$. Suppose that $h$ has a nonzero term $u^m$ $(m \ge 2)$, i.e., $h(u, 0) \ne 0$. By Theorem 3 in [3], the second order approximation of $h$ is $\phi_2(u, v) = -a^{-3}uv$. Let $p$ $(p > 2)$ denote the smallest order of nonzero term $u^m$. Comparing the coefficients in the relation

$$(M\phi_p)(u, v) = O(\|(u, v)\|^{p+1}), \tag{3.21}$$

we get $\gamma_{0,p} = 0$, a contradiction to the choice of $p$.

Next, we prove that the degree of $v$ in $h(u, v)$ is odd. For an indirect proof, assume that $h$ has a nonzero term like $u^iv^{2j}$ $(i \ge 0, j \ge 1)$. Let $q$ denote the smallest order $q = i + 2j > 2$. For any $k < q$ $(k \in \mathbb{Z}_+)$, the $k$-th order has the form of $u^{n_1}v^{2n_2+1}$, $n_1 \ge 0$, $n_2 \ge 0$, $k = n_1 + 2n_2 + 1$. Since $\gamma_{i,2j} \ne 0$, the $q$-th order approximation $\phi_q$ has a nonzero term $u^{\ell_1}v^{\ell_2}$ which satisfies one of the following

$$\begin{aligned} (\text{I}): &\quad i = n_1 + \ell_1 - 1, 2j = 2n_2 + \ell_2 + 2, \\ (\text{II}): &\quad i = n_1 + \ell_1 + 1, 2j = 2n_2 + \ell_2. \end{aligned}$$

From both cases (I) and (II) we obtain $\ell_2 \equiv 0 (\text{mod } 2)$, a contradiction to the fact that $q$ is the smallest order. Thus, $h(u, v)$ is of the form

$$\sum_{i+2j\ge 1, i,j\ge 0} \gamma_{i,2j+1}u^iv^{2j+1}$$

which implies that $h(u, -v) = -h(u, v)$. □

Lemma 1 enables us to give the following result about *singular center*.

**Theorem 5.** *When $b = c = 0$ and $a \ne 0$, $O$ is a singular center of system (1.2) on the 2-dimensional center manifold, which is orbitally equivalent to a center but crossed by the singular line $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0, x_2 = 0\}$.*

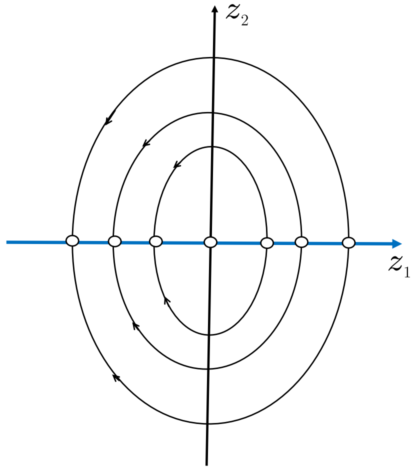


**Fig. 2.** Qualitative properties of $O$ for system (3.19) on the 2-dimensional center manifold.

**Proof.** Restricted to the 2-dimensional center manifold $\mathcal{W}^c$ given in (3.20), system (3.19) is presented as

$$\begin{cases} \frac{dx_1}{d\sigma_1} = 4x_2^2 + 4ax_2h(x_1, x_2), \\ \frac{dx_2}{d\sigma_1} = -x_1x_2 - ax_1h(x_1, x_2), \end{cases} \tag{3.22}$$

where $d\sigma_1 = a^{-2}d\tau_2$. With another time-rescaling $d\sigma_2 = x_2 d\sigma_1$, system (3.22) can be transformed into the form

$$\begin{cases} \frac{dx_1}{d\sigma_2} = 4x_2 + 4ah(x_1, x_2), \\ \frac{dx_2}{d\sigma_2} = -x_1 - ax_1h(x_1, x_2)/x_2. \end{cases} \tag{3.23}$$

System (3.23) has an equilibrium at $(0, 0)$ with eigenvalues $\pm 2\mathbf{i}$, i.e., the equilibrium $(0, 0)$ of system (3.23) is of center-type. Normalizing the linear part of system (3.23), we obtain

$$\begin{cases} \frac{dx_1}{d\sigma_2} = -x_2 + W_1(x_1, x_2), \\ \frac{dx_2}{d\sigma_2} = x_1 + W_2(x_1, x_2), \end{cases} \tag{3.24}$$

where $W_1(x_1, x_2) := ax_2h(2x_2, x_1)/x_1$ and $W_2(x_1, x_2) := ah(2x_2, x_1)$. By Lemma 1,

$$W_1(-x_1, x_2) = -ax_2h(2x_2, -x_1)/x_1 = ax_2h(2x_2, x_1)/x_1 = W_1(x_1, x_2),$$
$$W_2(-x_1, x_2) = ah(2x_2, -x_1) = -ah(2x_2, x_1) = -W_2(x_1, x_2),$$

implying that system (3.24) is time-reversible [24] with respect to the $x_2$-axis. Therefore, $(0, 0)$ is a center of system (3.24). This completes the proof. □

The so-called "singular center", obtained in Theorem 5, is actually surrounded by an infinite set of heteroclinic orbits on the 2-dimensional center manifold $\mathcal{W}^c$, each of which is heteroclinic to two equilibria on the singular line, as shown in Fig. 2. System (1.2) restricted to the center manifold is orbitally topologically equivalent to a center except for the singular line.

### *3.4. Case $a = b = c = 0$*

The last three cases in Theorem 3 intersect mutually at the point $(a, b, c) = (0, 0, 0)$, at which system (1.2) has two polynomial first integrals $2x_1^2 + x_3$ and $4x_2^2 + x_3^2$, which will be referred to cases **(3)** and **(4)**

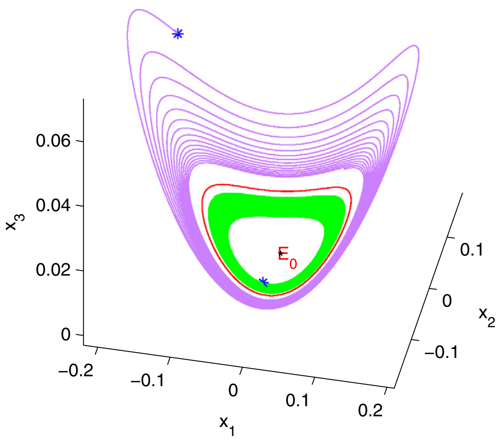


**Fig. 3.** Phase portraits of (1.2) for $a = -1$, $b = 1 - 5 \times 10^{-3}$, $c = 1$ and $\delta_0 = -2$.

of Theorem 3 respectively. Consequently, system (1.2) is integrable [2], which was also investigated in [23]. Actually, the solution of system (1.2) restricted to the invariant surface $2x_1^2 + x_3 = \nu_1$ ($\forall \nu_1 \geq 0$) is $4x_2^2 + (\nu_1 - 2x_1^2)^2 = \nu_2$ for all $\nu_2 \geq 0$. It implies that the equilibria $(\pm\sqrt{\nu_1/2}, 0, 0)$ are both centers.

## 4. Numerical simulations

In this section we make numerically simulations to illustrate one limit cycle produced from the Hopf bifurcation near $E_0$ (given in Subsection 2.1), three limit cycles from the Hopf bifurcation near $E_\pm$ (given in Subsection 2.2) and centers, which are not indicated in [15].

Consider system (1.2) with $a = -1$, $c = 1$ and $\delta_0 = -2$. Clearly, $(a, b, c, \delta_0) \in \mathcal{D}_0$. As done in Subsection 2.1 for weak focus $E_0$, from (2.2) and (2.9) we compute bifurcation parameter values

$$b = 1, \quad L_1 = -\frac{3}{20}.$$

Theorem 1 implies that a Hopf bifurcation happens as $\varepsilon = 0$. Simulation with the program "ode45" in MATLAB shows that for $\varepsilon = -5 \times 10^{-3} < 0$ a unique stable limit cycle (the thin cycle in Fig. 3) appears.

Consider system (1.2) with $c = 1$. Clearly, $(a, b, c, \delta_0) \in \mathcal{D}_3$. As done in Subsection 2.2 for weak focus $E_+$, from (2.10) and (2.12) we compute bifurcation parameter values

$$a = \zeta_1 \approx 7.1008009023 \times 10^{-3}, \quad b = B(\zeta_1, 1, \Xi(\zeta_1, 1)) \approx -7.0999737066 \times 10^{-1},$$
$$\delta_0 = \Xi(\zeta_1, 1) \approx 2.0013204743 \times 10^{-4}, \quad L_1 = L_2 = 0, \quad L_3 = 9.5324853736 \times 10^7.$$

Theorem 2 implies that a Hopf bifurcation with codimension 3 degeneracy happens. Simulation with the program "ode45" in MATLAB shows that for $\epsilon = -10^{-10}$, $\delta_0 = \Xi(\zeta_1, 1) + 4.6355 \times 10^{-8}$ and $a = \zeta_1 + 10^{-10}$ three limit cycles appear near $E_+$ (three thin cycles in Fig. 4). The three cycles of system (1.2) locating from inside to outside are unstable, stable and unstable respectively.

In order to simulate the center given in Subsection 3.2, Consider system (1.2) with $a = 1$, $b = -1$, $c = 2$ and $\delta_0 = -2$. Clearly, $(a, b, c, \delta_0) \in \mathcal{D}_0$ and $a = c/2$. Then $E_0$ is a center by Theorem 4. Choosing an initial point $P_0$ arbitrarily on the invariant surface $\mathcal{F}_1 : \ 2x_1^2 + x_3 - \delta_0 = 0$ but not at $E_0$, we use the MATLAB program "ode45" to simulate the orbit. For example, choosing $(0, 1, -2)$, $(1, 0, -4)$, $(1.5, 0, -6.5)$, $(2, -1, -10)$ and $(2.5, 0, -14.5)$, we observe in Fig. 5 that the five simulated orbits are all periodic orbits around $E_0$.

Finally, consider system (1.2) with $a = 1$, $b = -1$, $c = 2$ and $\delta_0 = 0$. Clearly, $(a, b, c, \delta_0) \in \mathcal{D}$ and $a = c/2$. Choosing an initial point $(1, 1, -2)$, which satisfies $2x_1^2 + x_3 - \delta_0 = 0$ (the invariant surface $\mathcal{F}_1$)

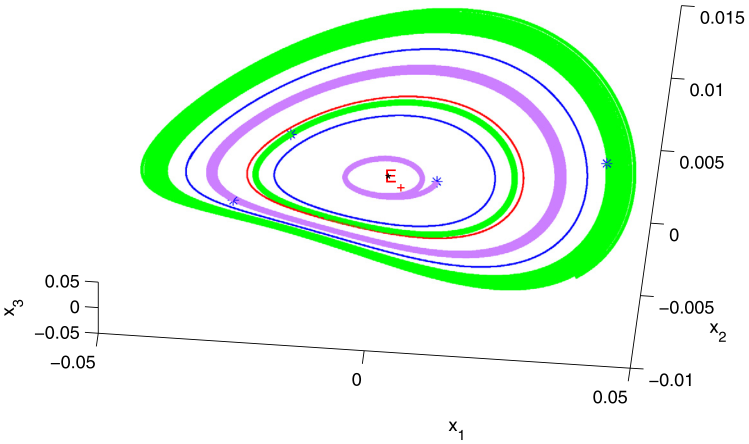


**Fig. 4.** Phase portraits of (1.2) for $a = \zeta_1 + 10^{-10}$, $b = B(\zeta_1, 1, \Xi(\zeta_1, 1)) - 10^{-10}$, $c = 1$ and $\delta_0 = \Xi(\zeta_1, 1) + 4.6355 \times 10^{-8}$.

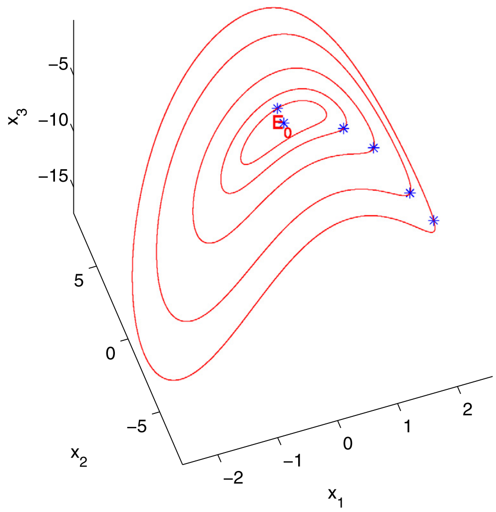


**Fig. 5.** Phase portraits of (1.2) for $a = 1$, $b = -1$, $c = 2$ and $\delta_0 = -2$.

and $y^2/2 + U(x) = 0$ (the Hamiltonian in the coordinate $(x, y, z)$), we use the MATLAB program "ode45" to simulate the orbit in the 3-dimensional phase space. As seen in Fig. 6, the orbit tends to the saddle $O$ as $t$ and $-t$ both increase, showing that the orbit is homoclinic to the saddle $O$. Similarly, from another initial point $(-1, -1, -2)$ we can plot the other homoclinic orbit. The two homoclinic orbits form a butterfly-shape. Further, choosing an initial point $P_1$ arbitrarily on the invariant surface $\mathcal{F}_1$ but outside the butterfly-shape of homoclinic orbits, we use the MATLAB program "ode45" to simulate the orbit. The initial point, e.g. $(0, 1, 0)$, can be found easily from the equality $2x_1^2 + x_3 - \delta_0 = 0$ and the inequality $y^2/2 + U(x) > 313/576$. We can observe from Fig. 6 that the orbit starting from $P_1$ is a periodic orbit. Similarly, choosing an initial point $P_2$ arbitrarily on the invariant surface $\mathcal{F}_1$ but inside one of the homoclinic orbits and not at $E_\pm$, we simulate the orbit. Such initial points, e.g. $(0.5, 0.5, -0.5)$, $(-0.5, -0.5, -0.5)$, $(0.3, 0.3, -0.18)$ and $(-0.3, -0.3, -0.18)$, can be found from the equality $2x_1^2 + x_3 - \delta_0 = 0$ and the inequality $0 < y^2/2 + U(x) < 313/576$. We can observe from Fig. 6 that all orbits starting from the initial points are also periodic orbits.

## Acknowledgments

The authors are grateful to the reviewer for his encouragement and helpful comments. Lingling Liu, Valery Romanovski and Weinian Zhang are also supported by a Marie Curie International Research Staff Exchange

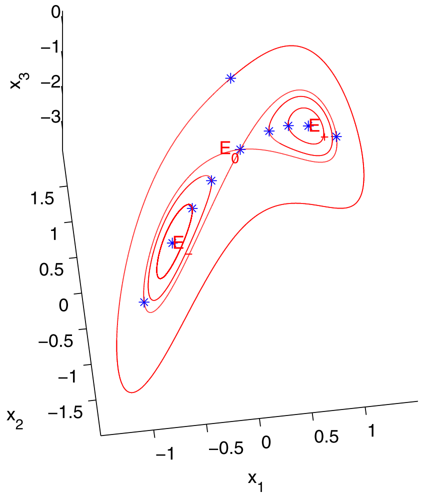


**Fig. 6.** Phase portraits of (1.2) for $a = 1$, $b = -1$, $c = 2$ and $\delta_0 = 0$.

Scheme Fellowship within the European Commission Seventh Framework Programme, FP7-PEOPLE-2012-IRSES-316338. Valery Romanovski acknowledges the support of the study by the Slovenian Research Agency Programme P1-0306, grants BI-CN/11-13-007, BI-TR/14-16-001 and thanks Dima Grigor'ev and Andreas Weber for very fruitful discussions on the topic. Weinian Zhang acknowledges the support by NSFC grant 11371264, 11221101 and 11231001. Lingling Liu acknowledges the support by young scholars development fund SWPU 201499010056.

## Appendix A

The functions $F_1$, $F_2$ and $F_3$ in (2.11) have the form

$$
\begin{aligned}
F_1 = {} & \{ac(\delta_0+a^2)(\delta_0+a^2+ac)(c\delta_0+4a^3+a^2c)^2[c\delta_0^3+a^2(4a+7c)\delta_0^2+a^4(8a+23c)\delta_0 \\
& +a^5(a+4c)(4a+c)]\}^{-1}\{a^3c^2(\delta_0+a^2)(2a-c)(c\delta_0+4a^3+a^2c)(\delta_0+3a^2)^3z_1^2 \\
& +2a^3\sqrt{ac(\delta_0+a^2)}(\delta_0+3a^2)[c^2\delta_0^4+c(8a^3+2a^2c+5c^2a-c^3)\delta_0^3+a^2(16a^4+24a^3c+12a^2c^2 \\
& +25c^3a-6c^4)\delta_0^2+a^4(32a^4+56a^3c+30a^2c^2+47c^3a-13c^4)\delta_0+a^6(a+c)(16a^3+24a^2c \\
& +27ac^2-8c^3)]z_1z_2+a^2c(\delta_0+3a^2)^2[c^2\delta_0+ac(8a^2+5ac+2c^2)\delta_0^3+a^3(16a^3+32a^2c \\
& +19ac^2+12c^3)\delta_0^2+a^5(32a^3+72a^2c+39ac^2+26c^3)\delta_0+16a^7(a+c)^3]z_2^2 \\
& +(1/2)\sqrt{ac(\delta_0+a^2)}(c\delta_0+4a^3+a^2c)(\delta_0+3a^2)[-c^2\delta_0^4+4a^2c(a-3c)\delta_0^3+2a^3(8a^3+2a^2c \\
& -17c^2a-2c^3)\delta_0^2+4a^5(a+c)(8a^2-5ac-4c^2)\delta_0+a^7(16a^3+12a^2c+3ac^2-20c^3)]z_1z_3 \\
& +(\delta_0+3a^2)(c\delta_0+4a^3+a^2c)[c^2\delta_0^5+ac(8a^2+3ac+2c^2)\delta_0^4+2a^3(8a^3+16a^2c+ac^2+5c^3)\delta_0^3 \\
& +2a^5(24a^3+32a^2c-5ac^2+5c^3)\delta_0^2+a^7(48a^3+64a^2c-19ac^2-26c^3)\delta_0+a^9(16a^3+24a^2c \\
& -9ac^2-44c^3)]z_2z_3-(1/4)(\delta_0+a^2)(c\delta_0+4a^3+a^2c)^2(\delta_0+3a^2)[3c\delta_0^3+a(8a^2+13ac+4c^2)\delta_0^2 \\
& +a^3(16a^2+29ac+16c^2)\delta_0+a^5(8a^2+19ac+20c^2)]z_3^2\},
\end{aligned}
$$

$$
\begin{aligned}
F_2 = {} & \{ac\sqrt{ac(\delta_0+a^2)}(\delta_0+a^2+ac)(c\delta_0+4a^3+a^2c)^2[c\delta_0^3+a^2(4a+7c)\delta_0^2+a^4(8a+23c)\delta_0 \\
& +a^5(a+4c)(4a+c)]\}^{-1}\{-a^3c(2a-c)(\delta_0+a^2)(\delta_0+3a^2)(c\delta_0+2a^3+2a^2c)[c\delta_0^2+4a^2(a+c)\delta_0 \\
& +a^3(4a^2+11ac-2c^2)]z_1^2+a^2c\sqrt{ac(\delta_0+a^2)}(\delta_0+3a^2)[c^2\delta_0^4+ac(4a^2+12ac-c^2)\delta_0^3 \\
& +a^3c(52a^2+38ac-5c^2)\delta_0^2+a^4(64a^4+76a^3c+108a^2c^2-35ac^3+4c^4)\delta_0+a^6(a+c)(64a^3+60a^2c
\end{aligned}
$$

$$
\begin{aligned}
&- 27ac^2 + 4c^3)]z_1z_2 - a^3c^2(\delta_0 + 3a^2)^2[-c(a-c)\delta_0^3 - a^2(4a^2 - 5ac - 3c^2)\delta_0^2 + a^3(16a^3 + a^2c \\
&+ 17ac^2 - 4c^3)\delta_0 + a^5(a+c)(20a^2 + 7ac - 4c^2)]z_2^2 + (1/2)(c\delta_0 + 4a^3 + a^2c)\sqrt{ac(\delta_0 + a^2)}[c^2\delta_0^5 \\
&+ a^2c(4a + 11c)\delta_0^4 + 2a^4c(20a + 23c)\delta_0^3 + 2a^5(16a^3 + 40a^2c + 65ac^2 - 4c^3)\delta_0^2 + a^6(64a^4 + 88a^3c \\
&+ 129a^2c^2 + 16ac^3 - 8c^4)\delta_0 + a^8(4a^2 - ac + 4c^2)(8a^2 + 13ac - 4c^2)]z_1z_3 - (1/2)ac(2a-c)(\delta_0 \\
&+ 3a^2)(c\delta_0 + 4a^3 + a^2c)[c\delta_0^4 + 2a^2(2a+c)\delta_0^3 - 4a^3(a-c)^2\delta_0^2 - 2a^6(10a + c)\delta_0 - 3a^7(4a^2 + 3ac \\
&- 4c^2)]z_2z_3 + (1/4)(\delta_0 + a^2)(c\delta_0 + 4a^3 + a^2c)^2[c\delta_0^4 + 2a(2a^2 + 2ac + c^2)\delta_0^3 + 2a^3(2a^2 + 13ac \\
&+ 2c^2)\delta_0^2 - 2a^4(2a^3 - 22a^2c - 11ac^2 + 4c^3)\delta_0 - a^6(4a^3 - 21a^2c - 36ac^2 + 16c^3)]z_3^2\},
\end{aligned}
$$

$$
\begin{aligned}
F_3 = {} & a(\delta_0 + 3a^2)\{\sqrt{ac(\delta_0 + a^2)}(\delta_0 + a^2)(c\delta_0 + 4a^3 + a^2c)^2[c\delta_0^3 + a^2(4a + 7c)\delta_0^2 + a^4(8a + 23c)\delta_0 \\
&+ a^5(a + 4c)(4a + c)]\}^{-1}\{-2a^3c(2a - c)(\delta_0 + a^2)(\delta_0 + 3a^2)(c\delta_0 + 2a^3 + 2a^2c)z_1^2 \\
&- 2a^2\sqrt{ac(\delta_0 + a^2)}[c^2\delta_0^3 - ac(4a^2 - 15ac + 2c^2)\delta_0^2 - a^3(a - 2c)(16a^2 + 16ac - 3c^2)\delta_0 \\
&- a^5(16a^3 - 36a^2c - 21ac^2 + 4c^3)]z_1z_2 + 2a^3c(\delta_0 + 3a^2)[c(3a - 2c)\delta_0^2 + 2a^2(a + c)(4a - 3c)\delta_0 \\
&+ a^4(8a^2 - 5ac - 4c^2)]z_2^2 + \sqrt{ac(\delta_0 + a^2)}(\delta_0 + a^2)(c\delta_0 + 4a^3 + a^2c)[c\delta_0^2 + 6a^2c\delta_0 + a^3(16a^2 - 7ac \\
&+ 4c^2)]z_1z_3 - 2a(2a - c)(\delta_0 + a^2)(c\delta_0 + 4a^3 + a^2c)[c\delta_0^2 + a^2(2a + c)\delta_0 + 2a^4(a - 2c)]z_2z_3 \\
&+ (1/2)(\delta_0 + a^2)^2(c\delta_0 + 4a^3 + a^2c)^2(\delta_0 - 5a^2 + 4ac)z_3^2\}.
\end{aligned}
$$

The function $\Omega_2$ in (2.12) has the form

$$
\begin{aligned}
&\Omega_2(a, c, \delta_0) \\
&\quad = 162c^5(3a + 4c)\delta_0^{20} + 9ac^4(78a^3 + 3173a^2c + 3366ac^2 - 396c^3)\delta_0^{19} + a^2c^3(4240a^5 \\
&\qquad + 33\,095a^4c + 779\,035a^3c^2 + 632\,136a^2c^3 - 158\,940ac^4 - 324c^5)\delta_0^{18} + a^4c^2(10\,608a^6 + 136\,524a^5c \\
&\qquad + 837\,950a^4c^2 + 12\,917\,089a^3c^3 + 7\,808\,294c^4a^2 - 3\,373\,596ac^5 - 13\,806c^6)\delta_0^{17} + a^5c(5760a^8 \\
&\qquad + 315\,456a^7c + 2\,191\,292a^6c^2 + 13\,524\,871a^5c^3 + 145\,589\,319a^4c^4 + 63\,248\,363a^3c^5 - 45\,503\,468a^2c^6 \\
&\qquad - 233\,244ac^7 - 990c^8)\delta_0^{16} + a^7(256a^9 + 148\,544a^8c + 4\,379\,968a^7c^2 + 23\,375\,112a^6c^3 + 148\,023\,240a^5c^4 \\
&\qquad + 1\,188v674\,618a^4c^5 + 344\,415\,926a^3c^6 - 437\,562\,968a^2c^7 - 2\,246\,450ac^8 + 6339c^9)\delta_0^{15} + a^8(5376a^{10} \\
&\qquad + 1\,737\,152a^9c + 37\,858\,720a^8c^2 + 182\,160\,872a^7c^3 + 1\,153\,386\,996a^6c^4 + 7\,319\,972\,926a^5c^5 \\
&\qquad + 1\,170\,089\,812a^4c^6 - 3\,184\,121\,498a^3c^7 - 14\,753\,835a^2c^8 + 712\,757ac^9 + 396c^{10})\delta_0^{14} + a^{10}(48\,896a^{10} \\
&\qquad + 12\,309\,952a^9c + 227\,714\,496a^8c^2 \sqrt[v]{} + 1\,074\,238\,328a^7c^3 + 6\,647\,472\,088a^6c^4 + 34\,919\,432\,930a^5c^5 \\
&\qquad + 1\,253\,403\,482a^4c^6 - 18\,150\,676\,862a^3c^7 - 80\,722\,138a^2c^8 + 13\,388\,285ac^9 - 32\,012c^{10})\delta_0^{13} \\
&\qquad + a^{12}(262\,912a^{10} + 59\,293\,760a^9c + 1\,007\,319\,648a^8c^2 + 4\,869\,143\,320a^7c^3 + 29\,138\,403\,284a^6c^4 \\
&\qquad + 131\,342\,566\,330a^5c^5 - 11\,794\,054\,172a^4c^6 - 82\,751\,022\,376a^3c^7 - 460\,655\,993a^2c^8 + 139\,892\,125ac^9 \\
&\qquad - 924\,031c^{10})\delta_0^{12} + a^{13}(948\,480a^{11} + 206\,069\,312a^{10}c + 3\,378\,744\,512a^9c^2 + 17\,120\,479\,848a^8c^3 \\
&\qquad + 99\,026\,035\,700a^7c^4 + 393\,900\,821\,812a^6c^5 - 88\,429\,860\,718a^5c^6 - 305v\,432\,042\,428a^4c^7 \\
&\qquad - 2\,766\,911\,666a^3c^8 + 979\,755\,941a^2c^9 - 11\,151\,274ac^{10} + 10\,532c^{11})\delta_0^{11} + a^{15}(2\,452\,736a^{11} \\
&\qquad + 535\,082\,944a^{10}c + 8\,746\,036\,640a^9c^2 + 46\,995\,421\,928a^8c^3 + 264\,167\,398\,666a^7c^4 + 947\,810\,455\,284a^6c^5 \\
&\qquad - 356\,100\,252\,020a^5c^6 - 918\,278\,669\,110a^4c^7 - 14\,699\,342\,547a^3c^8 + 4\,947\,716\,383a^2c^9 - 79\,621\,894ac^{10} \\
&\qquad + 123\,998c^{11})\delta_0^{10} + a^{17}(4\,722\,432a^{11} + 1\,060\,238\,784a^{10}c + 17\,646\,229\,920a^9c^2 + 101\,108\,214\,544a^8c^3 \\
&\qquad + 556\,950\,405\,572a^7c^4 + 1\,833\,743\,678\,052a^6c^5 - 1\,011\,496\,976\,314a^5c^6 - 2\,251\,845\,493\,082a^4c^7
\end{aligned}
$$

$$- 61\,875\,647\,382a^3c^8 + 18\,642\,517\,339a^2c^9 - 374\,762\,902ac^{10} + 195\,414c^{11})\delta_0^9 + a^{18}(6\,918\,912a^{12}$$
$$+ 1\,622\,432\,064a^{11}c + 27\,875\,915\,232a^{10}c^2 + 170\,699\,087\,376a^9c^3 + 929\,922\,315\,050a^8c^4$$
$$+ 2\,848\,096\,058\,876a^7c^5 - 2\,168\,275\,934\,934a^6c^6 - 4\,490\,994\,868\,220a^5c^7 - 199\,830\,454\,601a^4c^8$$
$$+ 53\,174\,412\,183a^3c^9 - 1\,207\,803\,097a^2c^{10} - 5\,207\,380ac^{11} + 25\,608c^{12})\delta_0^8 + a^{20}(7\,797\,504a^{12}$$
$$+ 1\,926\,640\,320a^{11}c + 34\,469\,383\,872a^{10}c^2 + 225\,663\,258\,008a^9c^3 + 1\,226\,553\,813\,064a^8c^4$$
$$+ 3\,532\,292\,054\,870a^7c^5 - 3\,585\,242\,022\,206a^6c^6 - 7\,232\,031\,398\,608a^5c^7 - 491\,961\,075\,398a^4c^8$$
$$+ 114\,951\,845\,813a^3c^9 - 2\,657\,129\,204a^2c^10 - 46\,634\,320ac^{11} + 321\,768c^{12})\delta_0^7 + a^{22}(6\,772\,480a^{12}$$
$$+ 1\,772\,158\,784a^{11}c + 33\,187\,481\,056a^{10}c^2 + 232\,112\,389\,560a^9c^3 + 1\,268\,641\,320\,756a^8c^4$$
$$+ 3\,464\,737\,737\,042a^7c^5 - 4\,586\,483\,877\,684a^6c^6 - 9\,291\,287\,384\,310a^5c^7 - 919v169\,411\,289a^4c^8$$
$$+ 186\,395\,461\,243a^3c^9 - 3\,784\,987\,480a^2c^{10} - 199\,877\,372ac^{11} + 1\,648\,632c^{12})\delta_0^6 + a^{23}(4\,502\,784a^{13}$$
$$+ 1\,251\,104\,576a^{12}c + 24\,596\,364\,736a^{11}c^2 + 183\,514\,882\,760a^{10}c^3 + 1\,015\,324\,958\,328a^9c^4$$
$$+ 2\,647\,375\,904\,046a^8c^5 - 4\,499\,079\,350\,034a^7c^6 - 9\,353\,568\,379\,458a^6c^7 - 1\,289\,831\,005\,518a^5c^8$$
$$+ 221\,502\,742\,011a^4c^9 - 2\,873\,886\,088a^3c^{10} - 518\,381\,756a^2c^{11} + 4\,021\,928ac^{12} + 14\,976c^{13})\delta_0^5$$
$$+ a^{25}(2\,253\,056a^{13} + 665\,444\,032a^{12}c + 13\,752\,775\,200a^{11}c^2 + 109\,263\,519\,272a^{10}c^3$$
$$+ 615\,240\,310\,228a^9c^4 + 1\,539\,657\,049\,248a^8c^5 - 3\,316\,954\,240\,212a^7c^6 - 7\,187\,260\,941\,408a^6c^7$$
$$- 1\,332\,266\,950\,467a^5c^8 + 185\,145\,265\,163a^4c^9 + 147\,659\,023a^3c^{10} - 856\,155\,840a^2c^{11} + 3\,377\,768ac^{12}$$
$$+ 114\,016c^{13})\delta_0^4 + a^{27}(822\,016a^{13} + 258\,091\,712a^{12}c + 5\,609\,238\,336a^{11}c^2 + 47\,357\,335\,288a^{10}c^3$$
$$+ 272\,546\,584\,382a^9c^4 + 657\,316\,294\,191a^8c^5 - 1\,775\,405\,025\,120a^7c^6 - 4\,053\,227\,753\,328a^6c^7$$
$$- 978\,567\,632\,982a^5c^8 + 101\,213\,346\,307a^4c^9 + 2\,488\,463\,582a^3c^{10} - 899\,418\,276a^2c^{11} - 4\,578\,824ac^{12}$$
$$+ 316\,256c^{13})\delta_0^3 + a^{28}(206\,592a^{14} + 68\,912\,960a^{13}c + 1\,574\,481\,632a^{12}c^2 + 14\,095\,815\,912a^{11}c^3$$
$$+ 83\,185\,003\,655a^{10}c^4 + 193\,961\,491\,169a^9c^5 - 650\,164\,392\,180a^8c^6 - 1\,575\,302\,734\,878a^7c^7$$
$$- 481\,793\,117\,541a^6c^8 + 31\,086\,463\,569a^5c^9 + 2\,170\,815\,122a^4c^{10} - 586\,187\,026a^3c^{11} - 12\,142\,296a^2c^{12}$$
$$+ 343\,136ac^{13} + 1664c^{14})\delta_0^2 + a^{30}(32\,000a^{14} + 11\,328\,832a^{13}c + 271\,890\,800a^{12}c^2 + 2\,575\,592\,804a^{11}c^3$$
$$+ 15\,623\,907\,502a^{10}c^4 + 35\,306\,694\,059a^9c^5 - 145\,469\,603\,508a^8c^6 - 375\,753\,549\,450a^7c^7$$
$$- 142\,060\,101\,972a^6c^8 + 3\,137\,100\,941a^5c^9 + 752\,505\,322a^4c^{10} - 221\,951\,434a^3c^{11} - 9\,760\,648a^2c^{12}$$
$$+ 134\,560ac^{13} + 1664c^{14})\delta_0 + 9a^{33}(256a^{13} + 96\,064a^{12}c + 2\,419\,296a^{11}c^2 + 24\,197\,716a^{10}c^3$$
$$+ 151\,149\,103a^9c^4 + 331\,644\,501a^8c^5 - 1\,664\,266\,629a^7c^6 - 4\,593\,869\,904a^6c^7 - 2\,101\,159\,719a^5c^8$$
$$- 49\,913\,273a^4c^9 + 8\,763\,265a^3c^{10} - 4\,319\,580a^2c^{11} - 322\,832ac^{12} - 320c^{13}).$$

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